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Vector Function: Let P be a variable point on a curve in space and the position vector of P relative to a fixed origin O be $\vec{r}$. If there exists an independent scalar variable t such that corresponding to each value of t in a definite domain, we get a definite position of P, that is, a unique vector $\vec{r}$, then $\vec{r}$ is called a single-valued vector function of the scalar variable t in that domain .It is usually denoted by $\vec{r} = \overrightarrow{f(t)}$.

$\overrightarrow{f(c)}$ denotes the particular vector for some fixed value c of t.

If $\vec{i}, \vec{j}, \vec{k}$ denote a fixed triad of mutually orthogonal unit vectors, then the vector function $\overrightarrow{f(t)}$ of the scalar parameter t can be decomposed to express if as in the form $\vec{r} = \overrightarrow{f(t)} = f_1(t)\,\vec{i} + f_2(t)\,\vec{j} + f_3(t)\,\vec{k}$ in which $f_1(t)$, $f_2(t)$, $f_3(t)$ are three scalar function of t.

The point P, whose Cartesian co-ordinates are (f_1, f_2, f_3), describes a certain curve as t varies and hence the function $\vec{f}$ represents a curve.

For example $\vec{r} = \overrightarrow{f(t)} = at\,\vec{i} + b(1-t)\,\vec{j}$ is the vector equation of the straight line $\dfrac{x}{a} + \dfrac{y}{b} = 1$, $\vec{r} = \overrightarrow{f(\alpha)} = a\cos\alpha\,\vec{i} + b\sin\alpha\,\vec{i} + 0\,\vec{k}$, α being a scalar variable, is the vector equation of an ellipse with $2a$ and $2b$ as the major and minor axes respectively.

Limit and continuity of Vector function:

A vector function $\overrightarrow{f(t)}$ of the scalar parameter t is said to tend to a limit $\vec{l}$ as t tends to t_0, if corresponding to any pre-assigned positive quantity ε, however small, we can find out another positive quantity δ, such that $\left|\overrightarrow{f(t)} - \vec{l}\right| < \in$, when $0 < |t - t_0| < \delta$.

This is expressed by writing $\lim\limits_{t\to t_0} \overrightarrow{f(t)} = \vec{l}$.

A vector function $\overrightarrow{f(t)}$ is said to be continuous at $t = t_0$, if $\lim\limits_{t\to t_0} \overrightarrow{f(t)}$ exists, is finite and is equal to $\overrightarrow{f(t_0)}$

If $\overrightarrow{f(t)}$ be continuous for every value of t in a domain, then it is said to be continuous in that domain.

Derivative of a vector:

The derivative of a vector function $\vec{a} = \overrightarrow{f(t)}$ is denoted by

$$\overrightarrow{f'(t)} = \frac{d\vec{a}}{dt} = \lim_{\Delta t\to 0} \frac{\overrightarrow{f(t+\Delta t)} - \overrightarrow{f(t)}}{\Delta t}$$

When this limit exists, $\vec{a}$ is said to be derivable or differentiable.

Space Curve :

If, in particular, $\overrightarrow{f(t)}$ be the position vector $\overrightarrow{r(t)}$ of any point (x, y, z) relative to a set of rectangular axes with the origin O, then we have

$\overrightarrow{r(t)} = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$.

As t changes, the terminal point $\vec{r}$ describes a space curve, having parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$.

Then $\dfrac{\Delta\vec{r}}{\Delta t} = \dfrac{\overrightarrow{r(t+\Delta t)} - \overrightarrow{r(t)}}{\Delta t}$

is a vector in the direction of $\Delta\vec{r}$. If the limit of $\dfrac{\Delta\vec{r}}{\Delta t}$ exists as $\Delta t \to 0$ and is equal to $\dfrac{d\vec{r}}{dt}$, then this limit will be a vector in the direction of the tangent to the space curve at (x, y, z) and will be given by

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

Note: The derivative of a constant vector is the zero vector. If t denotes the time, $\dfrac{d\vec{r}}{dt}$ represents the velocity $\vec{v}$ with which the terminal point of $\vec{r}$ describes the curve. Similarly, $\dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$ represents its acceleration $\vec{a}$ along the curve.

Differentiation formulae:

If $\vec{A}$, $\vec{B}$ and $\vec{C}$ differentiable vector functions of a scalar u, and ϕ is a differentiable scalar function of u, then

1. $\dfrac{d}{du}(\vec{A} + \vec{B}) = \dfrac{d\vec{A}}{du} + \dfrac{d\vec{B}}{du}$
2. $\dfrac{d}{du}(\vec{A}.\vec{B}) = \vec{A}.\dfrac{d\vec{B}}{du} + \dfrac{d\vec{A}}{du}.\vec{B}$
3. $\dfrac{d}{du}(\vec{A} \times \vec{B}) = \vec{A} \times \dfrac{d\vec{B}}{du} + \dfrac{d\vec{A}}{du} \times \vec{B}$
4. $\dfrac{d}{du}(\phi\vec{A}) = \phi\,\dfrac{d\vec{A}}{du} + \dfrac{d\phi}{du}\vec{A}$
5. $\dfrac{d}{du}(\vec{A} \times \vec{B} \times \vec{C}) = \vec{A}.\vec{B} \times \dfrac{d\vec{c}}{du} + \vec{A}.\dfrac{d\vec{B}}{du} \times \vec{C} + \dfrac{d\vec{A}}{du}.\vec{B} \times \vec{C}$
6. $\dfrac{d}{du}\{\vec{A} \times (\vec{B} \times \vec{C})\} = \vec{A} \times (\vec{B} \times \dfrac{d\vec{c}}{du}) + \vec{A} \times (\dfrac{d\vec{B}}{du} \times \vec{C}) + \dfrac{d\vec{A}}{du} \times (\vec{B} \times \vec{C})$

Theorem: If $\overrightarrow{F'(t)}$ exists at $t = t_0$, then $\overrightarrow{F(t)}$ is continuous at $t = t_0$

Proof: Let $\overrightarrow{F'(t)}$ exists at $t = t_0$.Then $\overrightarrow{F'(t_0)} = \lim\limits_{\Delta t \to 0} \dfrac{\overrightarrow{F(t_0 + \Delta t)} - \overrightarrow{F(t_0)}}{\Delta t}$ exists

Now, $\lim_{\Delta t\to 0} \left|\overrightarrow{F(t_0+\Delta t)}-\overrightarrow{F(t)}\right| = \lim_{\Delta t\to 0}\left[\Delta t\left\{\frac{\overrightarrow{F(t_0+\Delta t)}-\overrightarrow{F(t_0)}}{\Delta t}\right\}\right]$

$$=\lim_{\Delta t\to 0}(\Delta t)\lim_{\Delta t\to 0}\left[\frac{\overrightarrow{F(t_0+\Delta t)}-\overrightarrow{F(t_0)}}{\Delta t}\right]$$

$$=0\,\overrightarrow{F'(t_0)}=\vec{0}$$

Therefore, $\lim_{\Delta t\to 0}\overrightarrow{F(t_0+\Delta t)}=\overrightarrow{F(t_0)}$ and this shows that $\overrightarrow{F(t)}$ is continuous at $t=t_0$.

Converse: The converse of the above theorem is not always true.

e.g., $\overrightarrow{F(t)}=|t|\vec{i}$, is continuous at $t=0$ but not derivable there.

For, $\left|\overrightarrow{F(t)}-\overrightarrow{F(0)}\right|=\left||t|\vec{i}-\vec{0}\right|=|t|$

whence, $\lim_{t\to 0}\overrightarrow{F(t)}=\vec{0}=\overrightarrow{F(0)}$.

So that $\overrightarrow{F(t)}$ is continuous at $t=0$.

But $\frac{\overrightarrow{F(t)}-\overrightarrow{F(0)}}{t-0}=\frac{|t|\vec{i}}{t}$

So that the limit is $\vec{i}$ and $-\vec{i}$ according as t tends to zero through positive or through negative values. Hence $\overrightarrow{F'(0)}$ does not exist, since the limit is not unique.

Theorem: The necessary and sufficient condition for a vector function $\overrightarrow{f(t)}$ to be a constant is that $\frac{d}{dt}(\overrightarrow{f(t)})=\vec{0}$

Proof: If $\overrightarrow{f(t)}$ be a constant vector, then for every change h of the scalar variable t, $\overrightarrow{f(t+h)}-\overrightarrow{f(t)}=\vec{0}$

Hence $\frac{d\vec{f}}{dt}=\lim_{h\to 0}\frac{\overrightarrow{f(t+h)}-\overrightarrow{f(t)}}{h}=\vec{0}$

Thus the condition is necessary.

To prove that this condition is also sufficient, we assume that the derivatives of $\overrightarrow{f(t)}$ is zero vector.

Let us express $\overrightarrow{f(t)}$ as $\overrightarrow{f(t)}= f_1(t)\,\vec{i}+f_2(t)\,\vec{j}+f_3(t)\,\vec{k}$, in which $f_1(t), f_2(t), f_3(t)$ are three scalar functions of t.

Then $\frac{d\vec{f}}{dt}=\vec{0}=\frac{df_1}{dt}\vec{i}+\frac{df_2}{dt}\vec{j}+\frac{df_3}{dt}\vec{k}$

This implies $\frac{df_1}{dt}=\frac{df_2}{dt}=\frac{df_3}{dt}=0$ and hence the scalar functions $f_1(t), f_2(t), f_3(t)$ are constants. Hence $\overrightarrow{f(t)}$ is a constant vector.

Theorem: The necessary and sufficient condition for a vector function $\vec{r} = \overrightarrow{f(t)}$ to have a constant magnitude is that $\vec{f} \cdot \frac{d\vec{f}}{dt} = 0$

Proof: Let $\vec{r} = \overrightarrow{f(t)}$ be a vector function of a scalar variable t.

Let $\left|\overrightarrow{f(t)}\right|$=constant. Then $\overrightarrow{f(t)} \cdot \overrightarrow{f(t)} = \left|\overrightarrow{f(t)}\right|^2$=constant.

$\therefore \frac{d}{dt}\left(\overrightarrow{f(t)}.\overrightarrow{f(t)}\right) = 0$ or $\overrightarrow{f(t)}.\frac{d}{dt}\left(\overrightarrow{f(t)}\right) + \frac{d}{dt}\left(\overrightarrow{f(t)}\right).\overrightarrow{f(t)}$=0

or $2\,\overrightarrow{f(t)}.\frac{d}{dt}\left(\overrightarrow{f(t)}\right)$=0 or $\overrightarrow{f(t)}.\frac{d}{dt}\left(\overrightarrow{f(t)}\right)$=0

Therefore, the condition is necessary.

To prove that this condition is also sufficient, let $\overrightarrow{f(t)}$ be a vector function such that the condition $\vec{f}.\frac{d\vec{f}}{dt} = 0$ holds.

Then we have $2\vec{f}.\frac{d\vec{f}}{dt} = 0$ or $\vec{f}.\frac{d\vec{f}}{dt} + \frac{d\vec{f}}{dt}.\vec{f} = 0$ or, $\frac{d}{dt}\left(\overrightarrow{f(t)}.\overrightarrow{f(t)}\right) = 0$.

Therefore, $\left|\overrightarrow{f(t)}\right|^2 =$ constant or, $\left|\overrightarrow{f(t)}\right| =$ constant.

Note: If a vector function $\overrightarrow{f(t)}$ has a constant length, then $\overrightarrow{f(t)}$ and $\frac{d\vec{f}}{dt}$ are perpendicular.

Theorem : The necessary and sufficient condition for a vector $\vec{r} = \overrightarrow{f(t)}$ to have a constant direction is that $\vec{f} \times \frac{d\vec{f}}{dt} = \vec{0}$

Proof: Let $g(t)$ be the magnitude of $\overrightarrow{f(t)}$ and $\overrightarrow{F(t)}$ be a vector function in the direction of $\overrightarrow{f(t)}$ whose modulus is unity for all values of t, so that $\overrightarrow{f(t)} = g(t)\,\vec{F}$ and therefore $\frac{d\vec{f}}{dt} = g(t)\,\frac{d\vec{F}}{dt} + \frac{dg}{dt}\,\vec{F}$.

Thus we have, $\vec{f} \times \frac{d\vec{f}}{dt} = \vec{f} \times (\mathrm{g}\frac{d\vec{F}}{dt} + \frac{dg}{dt}\,\vec{F})$

$= \mathrm{g}\,\vec{F} \times (\mathrm{g}\frac{d\vec{F}}{dt} + \frac{dg}{dt}\,\vec{F})$

$= g^2\,\vec{F} \times \frac{d\vec{F}}{dt}$, since $\vec{F} \times \vec{F} = \vec{0}$ …………… (1)

Now, if the direction of $\overrightarrow{f(t)}$ be constant, then $\vec{F}$ is a constant vector. So we have $\frac{d\vec{F}}{dt} = \vec{0}$

Hence, from (1), in this case $\vec{f} \times \frac{d\vec{f}}{dt} = \vec{0}$

Thus the condition is necessary.

To prove that this condition is also sufficient, we assume that $\vec{f}\times\frac{d\vec{f}}{dt}=\vec{0}$

Then, from (1), we have $g^2\,\vec{F}\times\frac{d\vec{F}}{dt}=\vec{0}$ ………………. (2)

Since $g(t)$ is not always zero,

we have from (2), $\vec{F}\times\frac{d\vec{F}}{dt}=\vec{0}$ …………………. (3)

Now, $\vec{F}$ being the vector with unit (constant) modulus,

so, we have, $\vec{F}\cdot\frac{d\vec{F}}{dt}=0$ …………………….. (4)

From (3) and (4), we have $\frac{d\vec{F}}{dt}=\vec{0}$

This implies that $\vec{F}$ is a constant vector.

Hence $\overrightarrow{f(t)}$ has a constant direction.

Exercise1: If $\hat{a}$ is a unit vector in the direction of the vector $\vec{b}$ then show that $\hat{a}\times\frac{d\hat{a}}{dt}=\frac{\left(\vec{b}\times\frac{d\vec{b}}{dt}\right)}{\vec{b}.\vec{b}}$.

☺. Since $\hat{a}$ is a unit vector in the direction of the vector $\vec{b}$, therefore we have $\hat{a}=\frac{\vec{b}}{|\vec{b}|}$.

Now, $\frac{d\hat{a}}{dt}=\frac{1}{|\vec{b}|}\frac{d\vec{b}}{dt}-\frac{1}{|\vec{b}|^2}\frac{d|\vec{b}|}{dt}\vec{b}$

$$\hat{a}\times\frac{d\hat{a}}{dt}=\left(\vec{b}\times\frac{d\vec{b}}{dt}\right)\frac{1}{|\vec{b}|^2}-\frac{1}{|\vec{b}|^3}\frac{d|\vec{b}|}{dt}\left(\vec{b}\times\vec{b}\right)=\frac{\left(\vec{b}\times\frac{d\vec{b}}{dt}\right)}{\vec{b}.\vec{b}}$$, since $\left(\vec{b}\times\vec{b}\right)=\vec{0}$.

Exercise2: If $\vec{\omega}$ is a constant vector, $\vec{r}$ and $\vec{s}$ are vector functions of a scalar variable t and if $\frac{d\vec{r}}{dt}=\vec{\omega}\times\vec{r},\frac{d\vec{s}}{dt}=\vec{\omega}\times\vec{s}$ then show that $\frac{d}{dt}\left(\vec{r}\times\vec{s}\right)=\vec{\omega}\times\left(\vec{r}\times\vec{s}\right)$

☺. $\frac{d}{dt}\left(\vec{r}\times\vec{s}\right)=\frac{d\vec{r}}{dt}\times\vec{s}+\vec{r}\times\frac{d\vec{s}}{dt}=\left(\vec{\omega}\times\vec{r}\right)\times\vec{s}+\vec{r}\times\left(\vec{\omega}\times\vec{s}\right)$

$=\left(\vec{\omega}.\vec{s}\right)\vec{r}-\left(\vec{r}.\vec{s}\right)\vec{\omega}+\left(\vec{r}.\vec{s}\right)\vec{\omega}-\left(\vec{r}.\vec{\omega}\right)\vec{s}=\left(\vec{\omega}.\vec{s}\right)\vec{r}-\left(\vec{\omega}.\vec{r}\right)\vec{s}=\vec{\omega}\times\left(\vec{r}\times\vec{s}\right)$

Exercise3: A particle moves along a curve whose parametric equations are $x=e^{-t}$, $y=2\cos 3t$, $z=2\sin 3t$, where t is the time.

(a) Determine its velocity and acceleration at any time.

(b) Find the magnitudes of the velocity and acceleration at $t=0$.

☺. (a) The position vector $\vec{r}$ of the particle is

$$\vec{r}=x\,\vec{i}+y\,\vec{j}+z\,\vec{k}=e^{-t}\,\vec{i}+2\cos 3t\,\vec{j}+2\sin 3t\,\vec{k}$$

Then the velocity $\vec{v}=\frac{d\vec{r}}{dt}=-e^{-t}\ \vec{i}\ -6\sin 3t\ \vec{j}+6\cos 3t\ \vec{k}$ and the acceleration is $\vec{a}=\frac{d^2\vec{r}}{dt^2}=e^{-t}\vec{i}-18\cos 3t\vec{j}-18\sin 3t\vec{k}$

(b) At $t=0$, $\frac{d\vec{r}}{dt}=-\vec{i}+6\vec{k}$ and $\frac{d^2\vec{r}}{dt^2}=\vec{i}-18\vec{j}$

Then the magnitude of velocity at $t=0$ is $\sqrt{(-1)^2+6^2}=\sqrt{37}$

magnitude of acceleration at $t=0$ is $\sqrt{(1)^2+(-18)^2}=\sqrt{325}$

Exercise4: A particle moves along a curve $x=2t^2$, $y=t^2-4t$, $z=3t-5$, where t is the time. Find the components of its velocity and acceleration at time $t=1$ in the direction $\vec{i}-3\vec{j}+2\vec{k}$.

☺**.** The position vector $\vec{r}$ of the particle is

$$\vec{r}=x\ \vec{i}+y\ \vec{j}+z\ \vec{k}=2t^2\ \vec{i}+(t^2-4t)\ \vec{j}+(3t-5)\ \vec{k}$$

Then the velocity $\vec{v}=\frac{d\vec{r}}{dt}=4t\ \vec{i}+(2t-4)\ \vec{j}+3\vec{k}$

and the acceleration is $\vec{a}=\frac{d^2\vec{r}}{dt^2}=4\vec{i}+2\vec{j}$

At $t=1$, $\frac{d\vec{r}}{dt}=4\vec{i}-2\vec{j}+3\vec{k}$, $\frac{d^2\vec{r}}{dt^2}=4\vec{i}+2\vec{j}$

Unit vector in the direction of $\vec{i}-3\vec{j}+2\vec{k}$ is

$$\frac{\vec{i}-3\vec{j}+2\vec{k}}{\sqrt{1^2+(-3)^2+2^2}}=\frac{\vec{i}-3\vec{j}+2\vec{k}}{\sqrt{14}}$$

Then the component of the velocity in the given direction is

$$\frac{\left(4\vec{i}-2\vec{j}+3\vec{k}\right).\left(\vec{i}-3\vec{j}+2\vec{k}\right)}{\sqrt{14}}=\frac{4+6+6}{\sqrt{14}}=\frac{16}{\sqrt{14}}=\frac{8\sqrt{14}}{7}$$

and the component of the acceleration in the given direction is

$$\frac{(4\vec{i}+2\vec{j}).(\vec{i}-3\vec{j}+2\vec{k})}{\sqrt{14}}=\frac{4-6}{\sqrt{14}}=\frac{-2}{\sqrt{14}}=\frac{-\sqrt{14}}{7}$$

Exercise5: A particle moves so that its position vector is given by $\vec{r}=\cos\omega t\ \vec{i}+\sin\omega t\ \vec{j}$ where ω is a constant. Show that

(a) the velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$.

(b) the acceleration $\vec{a}$ is directed towards to the origin and has magnitude proportional to the distance from the origin,

(c) $\vec{r}\times\vec{v}$ =a constant vector.

☺**.** (a) $\vec{v}=\frac{d\vec{r}}{dt}=-\omega\sin\omega t\ \vec{i}+\omega\cos\omega t\ \vec{j}$

Then $\vec{r}.\vec{v}=(\cos\omega t\ \vec{i}+\sin\omega t\ \vec{j}).(-\omega\sin\omega t\ \vec{i}+\omega\cos\omega t\ \vec{j})$

$=(\cos\omega t)(-\omega\sin\omega t)+(\sin\omega t)(\omega\cos\omega t)=0$

Therefore, $\vec{r}$ and $\vec{v}$ are perpendicular.

(b) $\dfrac{d^2\vec{r}}{dt^2}=\dfrac{d\vec{v}}{dt}=-\omega^2\cos\omega t\ \vec{i}\ -\omega^2\sin\omega t\ \vec{j}$

$$=-\omega^2(\cos\omega t\ \vec{i}+\sin\omega t\ \vec{j})=-\omega^2\ \vec{r}$$

Then the acceleration is opposite to the direction of $\vec{r}$, i.e. it is directed toward the origin. Its magnitude is proportional to $|\vec{r}|$ which is the distance from the origin.

(c) $\vec{r}\times\vec{v}=(\cos\omega t\ \vec{i}+\sin\omega t\ \vec{j})\times(-\omega\sin\omega t\ \vec{i}+\omega\cos\omega t\ \vec{j})$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \cos wt & \sin wt & 0\\ -w\sin wt & w\cos wt & 0\end{vmatrix}$$

$=w(\cos^2\omega t+\sin^2\omega t)\vec{k}=\omega\ \vec{k}$, a constant vector.

Note: Physically, the motion is that of a particle moving on the circumference of a circle with constant angular speed ω. The acceleration, directed toward the centre of the circle, is the centripetal acceleration.

Exercise6: If $\vec{\alpha}=t^2\ \vec{i}\ -t\ \vec{j}+(2t+1)\ \vec{k}$ and $\vec{\beta}=(2t-3)\ \vec{i}+\vec{j}\ -t\ \vec{k}$, where $\vec{i}, \vec{j}, \vec{k}$ have their usual meanings, then $\dfrac{d}{dt}(\vec{\alpha}\times\dfrac{d\vec{\beta}}{dt})$ at $t=2$.

☺**.** We have $\dfrac{d}{dt}(\vec{\alpha}\times\dfrac{d\vec{\beta}}{dt})=\vec{\alpha}\times\dfrac{d^2\vec{\beta}}{dt^2}+\dfrac{d\vec{\alpha}}{dt}\times\dfrac{d\vec{\beta}}{dt}$

Now $\vec{\alpha}=t^2\ \vec{i}\ -t\ \vec{j}+(2t+1)\ \vec{k}$, $\dfrac{d\vec{\alpha}}{dt}=2t\ \vec{i}\ -\vec{j}\ +2\vec{k}$

$\vec{\beta}=(2t-3)\ \vec{i}+\vec{j}\ -t\ \vec{k}$, $\dfrac{d\vec{\beta}}{dt}=2\vec{i}\ -2\vec{k}$, $\dfrac{d^2\vec{\beta}}{dt^2}=\vec{0}$

$\therefore$ At $t=2$, $\vec{\alpha}=4\vec{i}\ -2\vec{j}\ +5\vec{k}$, $\dfrac{d\vec{\alpha}}{dt}=4\vec{i}\ -\vec{j}\ +2\vec{k}$

$\vec{\beta}=\vec{i}+\vec{j}\ -2\vec{k}$, $\dfrac{d\vec{\beta}}{dt}=2\vec{i}\ -\vec{k}$, $\dfrac{d^2\vec{\beta}}{dt^2}=\vec{0}$

$$\frac{d}{dt}(\vec{\alpha}\times\frac{d\vec{\beta}}{dt})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & -2 & 5\\ 0 & 0 & 0\end{vmatrix}+\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & -1 & 2\\ 2 & 0 & -1\end{vmatrix}=\vec{i}+8\vec{j}+2\vec{k}\ .$$