INEQUALITY

Introduction: An inequality is a mathematical reletion, which holds between two numbers. In this chepter we would like to discuss about the inequality of real numbers. Low of Trichotomi states that for each pair of real numbers a, b exactly one of the following relation holds:

a<b, a=b, a>b. Among this relations a<b and a>b are called inequalities. If a<b or a=b holds, then we write $a \le b$, which means that a is less than or equal to b. Similar result holds for $a \ge b$. If $a \le b$ and $a \ge b$, then by Low of Trichotomi a must be equal to b, i.e.,a=b.

We have some standard inequalities which have important role in the study of pure mathematics. Some of them are Holder`s inequality, Minkowskis inequality, Cauchy-Schwarz`s inequality, Weierstrass`s inequality etc.

Q1. If a,b,c be all real numbers prove that $a^2+b^2+c^2 \ge ab+bc+ca$.

☺. $a^2+b^2+c^2-(ab+bc+ca)=\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\} \ge 0$

Since each term is non negative. Therefore $a^2+b^2+c^2 \ge ab+bc+ca$, the equality occurs when $a=b=c$.

Q2. If a,b,c be all positive real numbers, prove that

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \ge a+b+c.$$

☺. $(a+b)^2+(a-b)^2=2(a^2+b^2)$

Therefore, $2(a^2+b^2) \ge (a+b)^2$, the equality occurs when $a=b$.

or, $\frac{a^2+b^2}{a+b} \ge \frac{a+b}{2}$, since $a+b>0$.

Similarly, $\frac{b^2+c^2}{b+c} \ge \frac{b+c}{2}$, $\frac{c^2+a^2}{c+a} \ge \frac{c+a}{2}$.

Hence, $\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \ge a+b+c$.

The equality occurs when $a=b=c$

Q3. If a,b,c,d be all real numbers greater than 1, prove that $(a+1)(b+1)(c+1)(d+1)<8(abcd+1)$.

☺. $(a-1)(b-1)>0$ since $a-1>0, b-1>0$.

or, $ab+1>a+b$

or, $2(ab+1)>ab+1+a+b=(a+1)(b+1)$

Therefore $(a+1)(b+1)<2(ab+1)$ …….(i)

Similarly, $(c+1)(d+1)<2(cd+1)$ …….(ii)

Now, $ab>1, cd>1$.

Using(i), we have $(ab+1)(cd+1)<2(abcd+1)$……(iii).

Hence, $(a+1)(b+1)(c+1)(d+1)<4(ab+1)(cd+1)$,by (i) and (ii);

and $4(ab+1)(cd+1)<8(abcd+1)$, by (iii)

Therefore $(a+1)(b+1)(c+1)(d+1)<8(abcd+1)$

Q4. If a,b,c,d be all real numbers less than 1, prove that $(a+1)(b+1)(c+1)(d+1)<8(abcd+1)$.

☺**.** $(a-1)(b-1)>0$ since $a-1<0, b-1<0$.

or, $ab+1>a+b$

or, $2(ab+1)>ab+1+a+b=(a+1)(b+1)$

Therefore $(a+1)(b+1)<2(ab+1)$ …(i)

Similarly, $(c+1)(d+1)<2(cd+1)$………(ii)

Now, $ab<1, cd<1$.

Using(i), we have $(ab+1)(cd+1)<2(abcd+1)$………(iii).

Hence, $(a+1)(b+1)(c+1)(d+1)<4(ab+1)(cd+1)$,by (i) and (ii);

and $4(ab+1)(cd+1)<8(abcd+1)$, by (iii)

Therefore $(a+1)(b+1)(c+1)(d+1)<8(abcd+1)$

Q5. If $a_1, a_2, ..., a_n$ be n positive real numbers in ascending order of magnitude, prove that $a_1<\frac{a_1^2+a_2^2+...+a_n^2}{a_1+a_2+...+a_n}<a_n$.

☺**.** Let $a_1, a_2, ..., a_n$ be n positive real numbers in ascending order of magnitude. Then we have $a_1<a_2<...<a_n$. …(1)

$a_1(a_1+a_2+...+a_n)=a_1^2+a_1a_2+a_1a_3+...+a_1a_n<a_1^2+a_2^2+...+a_n^2$ by (1).

Therefore $a_1<\frac{a_1^2+a_2^2+...+a_n^2}{a_1+a_2+...+a_n}$ …(2)

Again $a_n(a_1+a_2+...+a_n)=a_na_1+a_na_2+...+a_n^2>a_1^2+a_2^2+...+a_n^2$ by (1).

Therefore $\frac{a_1^2+a_2^2+...+a_n^2}{a_1+a_2+...+a_n}>a_n$ … (3)

Combining (2) and (3) we have, $a_1<\frac{a_1^2+a_2^2+...+a_n^2}{a_1+a_2+...+a_n}<a_n$.

Q6. If $a_1, a_2, ..., a_n$ be n positive real numbers, not all equal, and $p_1, p_2, ..., p_n$ be positive real numbers, prove that $\min(a)<\frac{p_1a_1+p_2a_2+...+p_na_n}{p_1+p_2+...+p_n}<\max(a)$.

☺**.** Let $a_1, a_2, ..., a_n$ be n positive real numbers, not all equal and $\min(a) = m$, $\max(a) = M$. Then he have $m \le a_i$ and $a_i \le M$ for $i = 1, 2, ..., n$. Not all of them are equalities.

$\min(a)(p_1 + p_2 + ... + p_n) = p_1 m + p_2 m + ... + p_n m < p_1 a_1 + p_2 a_2 + ... + p_n a_n$.

Therefore $\min(a) < \dfrac{p_1 a_1 + p_2 a_2 + ... + p_n a_n}{p_1 + p_2 + ... + p_n}$ …(1)

Again $\max(a)(p_1 + p_2 + ... + p_n) = p_1 M + p_2 M + ... + p_n M > p_1 a_1 + p_2 a_2 + ... + p_n a_n$.

Therefore $\dfrac{p_1 a_1 + p_2 a_2 + ... + p_n a_n}{p_1 + p_2 + ... + p_n} < \max(a)$ … (2)

Combining (1) and (2) we have, $\min(a) < \dfrac{p_1 a_1 + p_2 a_2 + ... + p_n a_n}{p_1 + p_2 + ... + p_n} < \max(a)$

Q7. If a, b, c be real numbers prove that $(a+b-c)^2 + (b+c-a)^2 + (c+a-b)^2 \ge ab + bc + ca$

☺: Clearly $(a+b-c)^2 + (b+c-a)^2 + (c+a-b)^2 = 3(a^2+b^2+c^2) - 2(ab+bc+ca)$

Again $(a^2+b^2+c^2) - (ab+bc+ca) = \dfrac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] \ge 0$

So , $(a^2+b^2+c^2) \ge (ab+bc+ca)$

Therefore $(a+b-c)^2 + (b+c-a)^2 + (c+a-b)^2 \ge 3(ab+bc+ca) - 2(ab+bc+ca)$

$\Rightarrow (a+b-c)^2 + (b+c-a)^2 + (c+a-b)^2 \ge ab + bc + ca$.

Q8. If a, b, c be positive real numbers such that the sum of any two is greater than the third, prove that $a^2(p-q)(p-r) + b^2(q-p)(q-r) + c^2(r-p)(r-q) \ge 0$ for all real p, q, r .

☺**.** Let a, b, c be positive real numbers such that the sum of any two is greater than the third. Then we have $a+b>c$, $b+c>a$, $c+a>b$. Therefore, $-c^2 > -(a+b)^2$.

$a^2(p-q)(p-r) + b^2(q-p)(q-r) + c^2(r-p)(r-q)$

$\ge a^2(p-r+r-q)(p-r) + b^2(q-r+r-p)(q-r) + (a+b)^2(r-p)(r-q)$

$= a^2(p-r)^2 + b^2(q-r)^2 + 2ab(p-r)(q-r)$

$= \{a(p-r) + b(q-r)\}^2 \ge 0$.

Therefore, $a^2(p-q)(p-r) + b^2(q-p)(q-r) + c^2(r-p)(r-q) \ge 0$.

Q9. If a, b, c be positive real numbers such that the sum of any two is greater than the third, prove that $a^2yz + b^2zx + c^2xy \le 0$ for all x, y, z such that $x+y+z=0$.

☺**.** Let a, b, c be positive real numbers such that the sum of any two is greater than the third. Then we have $a+b>c$, $b+c>a$, $c+a>b$.

$x+y+z=0 \Rightarrow z = -x-y$.

$a^2yz+b^2zx+c^2xy \le -\left(a^2y+b^2x\right)\left(x+y\right)+\left(a+b\right)^2 xy$

$$= -a^2y^2-b^2x^2+2abxy = -\left(ay-bx\right)^2 \le 0.$$

Therefore, $a^2yz+b^2zx+c^2xy \le 0$.

Q10. If n be a positive integer ≥ 3, prove that $\left(n!\right)^2 > n^n$.

☺**.** Let r be a positive integer such that $1<r<n$, $n \ge 3$.
Then $\left(r-1\right)\left(n-r\right)>0$ and therefore, we have $r\left(n-r+1\right)>n$.
Taking $r=2,3,...,n-1$ successively and multiplying, we have
$\left\{\left(n-1\right)!\right\}^2 > n^{n-2}$

Multiplying both sides by n^2 we have, $\left(n!\right)^2 > n^n$.

Q11. If a,b,c be the sides of a triangle and not all equal, show that

$$\frac{1}{2} < \frac{ab+bc+ca}{a^2+b^2+c^2} < 1.$$

☺**.** Since a,b,c are sides of a triangle, therefore we have
$a,b,c>0$ and $a+b>c, b+c>a, c+a>b$. From these we have
$c\left(a+b\right)+a\left(b+c\right)+b\left(c+a\right)>a^2+b^2+c^2$

i.e. $\dfrac{ab+bc+ca}{a^2+b^2+c^2} > \dfrac{1}{2}$ ……..(1)

Again, since a,b,c distinct real numbers. So, we have
$\left(a-b\right)^2+\left(b-c\right)^2+\left(c-a\right)^2>0$ i.e., $a^2+b^2+c^2>ab+bc+ca$

or, $\dfrac{ab+bc+ca}{a^2+b^2+c^2} < 1$ ……(2)

From (1) and (2) we have,

$$\frac{1}{2} < \frac{ab+bc+ca}{a^2+b^2+c^2} < 1.$$

Q12. If a,b,c be all positive real numbers, not all equal, prove that

(i) $2\left(a^3+b^3+c^3\right) > a^2\left(b+c\right)+b^2\left(c+a\right)+c^2\left(a+b\right) > 6abc$;

(ii) $\dfrac{b+c}{b^2+c^2}+\dfrac{c+a}{c^2+a^2}+\dfrac{a+b}{a^2+b^2} < \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$;

(iii) $\dfrac{bc}{b+c}+\dfrac{ca}{c+a}+\dfrac{ab}{a+b} < \dfrac{a+b+c}{2}$.

☺: Let a,b,c be all positive real numbers, not all equal.

(i) Clearly $a\left(a-b\right)^2 \ge 0$ or, $a^3+ab^2 \ge 2a^2b$

Similarly, $a^3+ac^2 \ge 2a^2c$.
Adding we have, $2a^3+ab^2+ac^2 \ge 2a^2\left(b+c\right)$ …(1)

Similarly, $2b^3+a^2b+bc^2 \ge 2b^2\left(c+a\right)$ …(2)

And $2c^3 + a^2c + b^2c \geq 2c^2(a+b)$ …(3)

Since a,b,c are not all equal, not all of them are equalities.

Adding (1), (2) and (3) we have

$$2(a^3+b^3+c^3)+a^2(b+c)+b^2(c+a)+c^2(a+b) > 2a^2(b+c)+2b^2(c+a)+2c^2(a+b)$$

Or, $2(a^3+b^3+c^3) > a^2(b+c)+b^2(c+a)+c^2(a+b)$ …. (4)

Again $(a-b)^2 \geq 0 \Rightarrow a^2+b^2 \geq 2ab \Rightarrow a^2c+b^2c \geq 2abc$

Similarly , $b^2a+c^2a \geq 2abc$ and $c^2b+a^2b \geq 2abc$

Since a,b,c are not all equal, not all of them are equalities.

Adding we get $a^2(b+c)+b^2(c+a)+c^2(a+b) > 6abc$ … (5)

Combining (4) and (5) we get

$$2(a^3+b^3+c^3) > a^2(b+c)+b^2(c+a)+c^2(a+b) > 6abc;$$

(ii) Clearly $(a-b)^2 \geq 0$.

$$(a-b)^2 \geq 0 \Rightarrow a^2+b^2 \geq 2ab \Rightarrow \frac{a+b}{a^2+b^2} \leq \frac{a+b}{2ab} \Rightarrow \frac{a+b}{a^2+b^2} \leq \frac{1}{2b}+\frac{1}{2a}$$

Similarly , $\frac{b+c}{b^2+c^2} \leq \frac{1}{2b}+\frac{1}{2c}$; and $\frac{c+a}{c^2+a^2} \leq \frac{1}{2c}+\frac{1}{2a}$.

Since a,b,c are not all equal, not all of them are equalities.

Adding we get $\frac{b+c}{b^2+c^2}+\frac{c+a}{c^2+a^2}+\frac{a+b}{a^2+b^2} < \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

(iii) We know that

$$4ab = (a+b)^2-(a-b)^2 \Rightarrow 4ab \leq (a+b)^2 \Rightarrow \frac{ab}{a+b} \leq \frac{a+b}{4}$$

Similarly $\frac{bc}{b+c} \leq \frac{b+c}{4}$ and $\frac{ca}{c+a} \leq \frac{c+a}{4}$.

Since a,b,c are not all equal, not all of them are equalities.

Adding we get $\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b} < \frac{a+b+c}{2}$.

Weierstrass's Ineuality: If $a_1, a_2, ..., a_n$ are all positive real numbers less than 1 and $s_n = a_1+a_2+...+a_n$, then

$$1-s_n < (1-a_1)(1-a_2)...(1-a_n) < \frac{1}{1+s_n}$$

And $1+s_n < (1+a_1)(1+a_2)...(1+a_n) < \frac{1}{1-s_n}$

Proof: $(1-a_1)(1-a_2) = 1-(a_1+a_2)+a_1a_2$

$> 1-(a_1+a_2)$

Therefore $(1-a_1)(1-a_2)(1-a_3) > [1-(a_1+a_2)](1-a_3)$ since $1-a_3 > 0$

$> 1-(a_1+a_2+a_3)$

Continuing this process upto n times we get
$(1-a_1)(1-a_2)...(1-a_n)>1-(a_1+a_2+...+a_n)$

i.e. $>1-s_n$.......................(i)

In the same manner, $(1+a_1)(1+a_2)...(1+a_n)>1+s_n$ ....(ii)

Again $1-a_1^2<1$

Therefore $1-a_1<\dfrac{1}{1+a_1}$, since $1+a_1>0$

Similarly $1-a_2<\dfrac{1}{1+a_2},...,1-a_n<\dfrac{1}{1+a_n}$

Therefore $(1-a_1)(1-a_2)...(1-a_n)<\dfrac{1}{(1+a_1)(1+a_2)...(1+a_n)}<\dfrac{1}{1+s_n}$ ....(iii)

In the same manner

$(1+a_1)(1+a_2)...(1+a_n)<\dfrac{1}{(1-a_1)(1-a_2)...(1-a_n)}<\dfrac{1}{1-s_n}$.......................$(iv)$ compiling

(i) and (iii) we get $1-s_n<(1-a_1)(1-a_2)...(1-a_n)<\dfrac{1}{1+s_n}$

compiling (ii) and (iv) we get $1+s_n<(1+a_1)(1+a_2)...(1+a_n)<\dfrac{1}{1-s_n}$.

Cauchy – Schwarz Inequality:

If $a_1,a_2,...,a_n;b_1,b_2,...,b_n$ be all real numbers, then
$\left(a_1^2+a_2^2+...+a_n^2\right)\left(b_1^2+b_2^2+...+b_n^2\right)\geq\left(a_1b_1+a_2b_2+...+a_nb_n\right)^2$, the equality occurs when either ,

(i) $a_i=0$ for $i=1,2,...,n;$ or $b_i=0$ for $i=1,2,...,n;$ or both $a_i=0$ and $b_i=0$ for $i=1,2,...,n;$

or, (ii) $a_i=kb_i$ for some non zero real $k, i=1,2,...,n.$

Proof: Case1: If $a_i=0$ for $i=1,2,...,n$; or $b_i=0$ for $i=1,2,...,n$; or both $a_i=0$ and $b_i=0$ for $i=1,2,...,n$; then the equality holds, each side being zero.

Case2: Let not all of a_i and not all of b_i be zero.

Sub-case1: Let $a_i=kb_i$ for some non zero real $k, i=1,2,...,n$

Then $\left(a_1^2+a_2^2+...+a_n^2\right)\left(b_1^2+b_2^2+...+b_n^2\right)=k^2\left(b_1^2+b_2^2+...+b_n^2\right)^2$

and $\left(a_1b_1+a_2b_2+...+a_nb_n\right)^2=k^2\left(b_1^2+b_2^2+...+b_n^2\right)^2$

Therefore the equality holds in this sub case.

Sub-case2: Let $(a_1,a_2,...,a_n)$ and $(b_1,b_2,...,b_n)$ be not proportional.

Let us consider the expression.

$(a_1-\lambda b_1)^2+(a_2-\lambda b_2)^2+...+(a_n-\lambda b_n)^2$, where λ is real.

For all real λ , the expression ≥ 0. The equality occurs only when $a_1-\lambda b_1=0, a_2-\lambda b_2=0, \ldots, a_n-\lambda b_n=0$

i.e, when $a_i=\lambda b_i$, $i=1,2,\ldots,n.$

i.e, when $a_1,a_2,\ldots,a_n;b_1,b_2,\ldots,b_n.$ are proportional.

Therefore, in this sub-case

$$\left(a_1-\lambda b_1\right)^2+\left(a_2-\lambda b_2\right)^2+\ldots+\left(a_n-\lambda b_n\right)^2>0 \quad \text{for all real } \lambda$$

or, $\left(a_1+a_2+\ldots+a_n\right)^2-2\lambda\left(a_1b_1+a_2b_2+\ldots+a_nb_n\right)+\lambda^2\left(b_1{}^2+b_2{}^2+\ldots+b_n{}^2\right)>0$

or, $B\lambda^2-2C\lambda+A>0$, where $A=a_1{}^2+a_2{}^2+\ldots+a_n{}^2$

$$B=b_1{}^2+b_2{}^2+\ldots+b_n{}^2$$

$$C=a_1b_1+a_2b_2+\ldots+a_nb_n$$

Therefore the roots of the equation $Bx^2-2Cx+A=0$ must be imaginary , because otherwise, there would exist some real λ for which the equality $B\lambda^2-2C\lambda+A=0$ would hold , a contradiction.

Therefore $AB>C^2$

or, $\left(a_1{}^2+a_2{}^2+\ldots+a_n{}^2\right)\left(b_1{}^2+b_2{}^2+\ldots+b_n{}^2\right)>\left(a_1b_1+a_2b_2+\ldots+a_nb_n\right)^2$

Note: (i) In particular, if $a_1,a_2,\ldots,a_n;b_1,b_2,\ldots,b_n$ be all positive real numbers, then $\left(a_1{}^2+a_2{}^2+\ldots+a_n{}^2\right)\left(b_1{}^2+b_2{}^2+\ldots+b_n{}^2\right)\geq\left(a_1b_1+a_2b_2+\ldots+a_nb_n\right)^2$, the equality occurs when $\dfrac{a_1}{b_1}=\dfrac{a_2}{b_2}=\ldots=\dfrac{a_n}{b_n}$.

(ii)If $\{x_1,x_2,\ldots,x_n\}$ and $\{y_1,y_2,\ldots,y_n\}$ be two sets of positive real numbers, prove that $\begin{vmatrix}\sum_{i=1}^{n}x_i^2 & \sum_{i=1}^{n}x_iy_i\\ \sum_{i=1}^{n}x_iy_i & \sum_{i=1}^{n}y_i^2\end{vmatrix}\geq 0$.

Q13. If a,b,c,x,y,z be all real numbers and $a^2+b^2+c^2=1$, $x^2+y^2+z^2=1$, prove that $-1\leq ax+by+cz\leq 1$.

☺**:** Let a,b,c,x,y,z be all real numbers such that $a^2+b^2+c^2=1$, $x^2+y^2+z^2=1$.

By Cauchy-Schwarz inequality we have, $(ax+by+cz)^2\leq\left(a^2+b^2+c^2\right)\left(x^2+y^2+z^2\right)$.

That is, $(ax+by+cz)^2\leq 1$ or, $-1\leq ax+by+cz\leq 1$.

Q14. If $a_1,a_2,\ldots,a_n;b_1,b_2,\ldots,b_n;c_1,c_2,\ldots,c_n$ be all positive real numbers prove that $\left(a_1b_1c_1+a_2b_2c_2+\ldots+a_nb_nc_n\right)^2<\left(a_1{}^2+a_2{}^2+\ldots+a_n{}^2\right)\left(b_1{}^2+b_2{}^2+\ldots+b_n{}^2\right)\left(c_1{}^2+c_2{}^2+\ldots+c_n{}^2\right)$

☺**.** Let $d_i=b_ic_i, i=1,2,\ldots,n$.

Then by Cauchy-Schwarz inequality we have,

$$\left(a_1d_1+a_2d_2+\ldots+a_nd_n\right)^2\leq\left(a_1{}^2+a_2{}^2+\ldots+a_n{}^2\right)\left(d_1{}^2+d_2{}^2+\ldots+d_n{}^2\right)$$

$$\left(b_1^2+b_2^2+...+b_n^2\right)\left(c_1^2+c_2^2+...+c_n^2\right)=\left(b_1^2c_1^2+b_2^2c_2^2+...+b_n^2c_n^2\right)+\left(b_1^2c_2^2+b_2^2c_1^2+...\right)$$
$$>\left(b_1^2c_1^2+b_2^2c_2^2+...+b_n^2c_n^2\right) \text{ since the 2}^{\text{nd}}\text{ term is positive}$$
$$=\left(d_1^2+d_2^2+...+d_n^2\right)$$

Therefore, $\left(d_1^2+d_2^2+...+d_n^2\right)=b_1^2c_1^2+b_2^2c_2^2+...+b_n^2c_n^2$

$$<\left(b_1^2+b_2^2+...+b_n^2\right)\left(c_1^2+c_2^2+...+c_n^2\right)$$ since b_i, c_i are all positive.

$$\therefore \left(a_1b_1c_1+a_2b_2c_2+...+a_nb_nc_n\right)^2<\left(a_1^2+a_2^2+...+a_n^2\right)\left(b_1^2+b_2^2+...+b_n^2\right)\left(c_1^2+c_2^2+...+c_n^2\right)$$

Q15. If a,b,c,d are four positive real numbers, then prove that

$$\left(a^4+1\right)\left(b^4+1\right)\left(c^4+1\right)\left(d^4+1\right)\geq\left(abcd+1\right)^4$$

☺. Since $a^2, b^2, 1$ are real numbers, therefore, by Cauchy's inequality we have,

$\left\{\left(a^2\right)^2+1^2\right\}\left\{\left(b^2\right)^2+1^2\right\}\geq\left(a^2b^2+1\right)^2$ i.e., $\left(a^4+1\right)\left(b^4+1\right)\geq\left(a^2b^2+1\right)^2$ ……….(1)

Similarly, $\left(c^4+1\right)\left(d^4+1\right)\geq\left(c^2d^2+1\right)^2$ …………………(2)

Combining (1) and (2) we have, $\left(a^4+1\right)\left(b^4+1\right)\left(c^4+1\right)\left(d^4+1\right)\geq\left(a^2b^2+1\right)^2\left(c^2d^2+1\right)^2$

Therefore, $\left(a^4+1\right)\left(b^4+1\right)\left(c^4+1\right)\left(d^4+1\right)\geq\left\{\left(a^2b^2+1\right)\left(c^2d^2+1\right)\right\}^2$ …………(3)

Again, by Cauchy's inequality we have, $\left\{\left(ab\right)^2+1\right\}\left\{\left(cd\right)^2+1\right\}\geq\left(abcd+1\right)^2$ ……(4)

Using (4) in (3) we have, $\left(a^4+1\right)\left(b^4+1\right)\left(c^4+1\right)\left(d^4+1\right)\geq\left\{\left(abcd+1\right)^2\right\}^2=\left(abcd+1\right)^4$.

Q16. If $a_1,a_2,a_3;b_1,b_2,b_3;c_1,c_2,c_3;d_1,d_2,d_3$ be all real numbers, prove that

$$\left(a_1b_1c_1d_1+a_2b_2c_2d_2+a_3b_3c_3d_3\right)^4\leq\left(a_1^4+a_2^4+a_3^4\right)\left(b_1^4+b_2^4+b_3^4\right)\left(c_1^4+c_2^4+c_3^4\right)\left(d_1^4+d_2^4+d_3^4\right).$$

Q17: If a,b,c,d be all real numbers, prove that

$$\left(a^2+b^2+c^2+d^2\right)\left(a^2+b^2\right)\left(c^2+d^2\right)\geq\left(abc+bcd+cda+dab\right)^2$$

☺: $\left(a^2+b^2+c^2+d^2\right)\left(a^2+b^2\right)\left(c^2+d^2\right)=\left(a^2+b^2+c^2+d^2\right)\left(a^2c^2+b^2c^2+a^2d^2+b^2d^2\right)$

Then by Cauchy-Schwarz inequality

$$\left(a^2+b^2+c^2+d^2\right)\left(b^2c^2+a^2d^2+b^2d^2+a^2c^2\right)\geq\left(abc+bad+cbd+dac\right)^2$$

Therefore $\left(a^2+b^2+c^2+d^2\right)\left(a^2+b^2\right)\left(c^2+d^2\right)\geq\left(abc+bcd+cda+dab\right)^2$.

Q18. Prove that the minimum value of $x^2+y^2+z^2$ is $\left(\frac{c}{7}\right)^2$ where x,y,z are positive real numbers subject to the condition $2x+3y+6z=c$, c being a constant. Find the values of x,y,z for which the minimum value is attained.

☺. Since, x,y,z are positive real numbers, therefore, by Cauchy-Schwarz's inequality we have, $\left(2^2+3^2+6^2\right)\left(x^2+y^2+z^2\right)\geq\left(2x+3y+6z\right)^2$, the equality occurs only when $\frac{x}{2}=\frac{y}{3}=\frac{z}{6}$.

$\because\ 2x+3y+6z=c,\ \therefore\ x^2+y^2+z^2 \geq \left(\frac{c}{7}\right)^2$. This shows that the minimum value of $x^2+y^2+z^2$ is $\left(\frac{c}{7}\right)^2$ and this minimum value is attained only when $\frac{x}{2}=\frac{y}{3}=\frac{z}{6}=k$ (say).

Now, from $2x+3y+6z=c$ we have, $49k=c \Rightarrow k=\frac{c}{49}$.Thus the minimum value is attained when $x=\frac{2c}{49}, y=\frac{3c}{49}, z=\frac{6c}{49}$

Q19. Find the minimum value of $x^2+y^2+z^2$ where x,y,z are positive real numbers subject to the condition $2x+3y+4z=c$, c being a constant. Find the values of x,y,z for which the minimum value is attained.

☺**.** Since x,y,z are positive real numbers, therefore, by Cauchy-Schwarz's inequality we have, $\left(2^2+3^2+4^2\right)\left(x^2+y^2+z^2\right) \geq \left(2x+3y+4z\right)^2$, the equality occurs only when $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$.

Since $2x+3y+4z=c$, $x^2+y^2+z^2 \geq \frac{c^2}{29}$. This shows that the minimum value of $x^2+y^2+z^2$ is $\frac{c^2}{29}$ and this minimum value is attained only when $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}=k$ (say).

Now, from $2x+3y+4z=c$ we have, $29k=c \Rightarrow k=\frac{c}{29}$.Thus the minimum value is attained when $x=\frac{2c}{29}, y=\frac{3c}{29}, z=\frac{6c}{29}$.

Tchebychev's Inequality: If $a_1,a_2,...,a_n;b_1,b_2,...,b_n$ be all real numbers, then

$$(i)\frac{a_1+a_2+...+a_n}{n}.\frac{b_1+b_2+...+b_n}{n} \leq \frac{a_1b_1+a_2b_2+...+a_nb_n}{n}$$

If $a_1 \leq a_2 \leq ... \leq a_n$ and $b_1 \leq b_2 \leq ... \leq b_n$;

Or, if $a_1 \geq a_2 \geq ... \geq a_n$ and $b_1 \geq b_2 \geq ... \geq b_n$

$$(ii)\frac{a_1+a_2+...+a_n}{n}.\frac{b_1+b_2+...+b_n}{n} \geq \frac{a_1b_1+a_2b_2+...+a_nb_n}{n}$$

If $a_1 \leq a_2 \leq ... \leq a_n$ and $b_1 \geq b_2 \geq ... \geq b_n$;

Or, if $a_1 \geq a_2 \geq ... \geq a_n$ and $b_1 \leq b_2 \leq ... \leq b_n$;

The equality occurs if either all a_i's are equal or all b_i's are equal.

Theorem: If $a_1,a_2,...,a_n$ be n positive real numbers, not all equal, and p,q are rational numbers, then $\frac{a_1^{p+q}+a_2^{p+q}+...+a_n^{p+q}}{n} > or < \frac{a_1^p+a_2^p+...+a_n^p}{n}.\frac{a_1^q+a_2^q+...+a_n^q}{n}$

according as p and q has same or opposite signs.

Q20: If a,b,c,d are all positive real numbers, not all equal, prove that

$$a^5+b^5+c^5+d^5 > abcd\left(a+b+c+d\right)$$

☺ : We have $\frac{a^{p+q}+b^{p+q}+c^{p+q}+d^{p+q}}{4} > \frac{a^p+b^p+c^p+d^p}{4}\frac{a^q+b^q+c^q+d^q}{4}$ if p,q are rational numbers of the same sign.

Taking $p=4, q=1$ we get $\frac{a^5+b^5+c^5+d^5}{4} > \frac{a^4+b^4+c^4+d^4}{4}.\frac{a+b+c+d}{4}$

But $\frac{a^4+b^4+c^4+d^4}{4} > \sqrt[4]{\left(a^4b^4c^4d^4\right)} = abcd$

Therefore $a^5+b^5+c^5+d^5 > abcd\left(a+b+c+d\right)$

Arithmetic, Geometric and Harmonic Means:

Let $a_1, a_2, ..., a_n$ be n positive real numbers.

The arithmetic mean (A.M.) of the numbers are defined by $\frac{a_1+a_2+...+a_n}{n}$ and is denoted by A.

The geometric mean(G.M) of the numbers is defined by $\sqrt[n]{a_1a_2...a_n}$ and is denoted by G.

The Harmonic mean(H.M) of the numbers is defined by $\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}}$ and is denoted by H.

Theorem: If $a_1, a_2, ..., a_n$ be n positive real numbers and A, G, H be their arithmetic mean, geometric mean, harmonic mean respectively then $A \ge G \ge H$.

Hence prove that $\left(1+A\right)^n \ge \left(1+a_1\right)\left(1+a_2\right)..........\left(1+a_n\right) \ge \left(1+G\right)^n$

Proof: $a_1a_2 = \left(\frac{a_1+a_2}{2}\right)^2 - \left(\frac{a_1-a_2}{2}\right)^2 \le \left(\frac{a_1+a_2}{2}\right)^2$, since $\left(\frac{a_1-a_2}{2}\right)^2 \ge 0$ ……..(i)

The equality occurs when $\left(\frac{a_1-a_2}{2}\right)^2 = 0$ i.e, when $a_1 = a_2$

Similarly, $a_3a_4 \le \left(\frac{a_3+a_4}{2}\right)^2$, the equality occurs when $a_3 = a_4$

Therefore $a_1a_2a_3a_4 \le \left(\frac{a_1+a_2}{2}\right)^2\left(\frac{a_3+a_4}{2}\right)^2$ ………………..(ii)

But $\frac{a_1+a_2}{2}\cdot\frac{a_3+a_4}{2} \le \left(\frac{a_1+a_2+a_3+a_4}{4}\right)^2$ from (i)

The equality occurs when $\frac{a_1+a_2}{2} = \frac{a_3+a_4}{2}$. Therefore

$$a_1a_2a_3a_4 \le \left(\frac{a_1+a_2+a_3+a_4}{4}\right)^4 \text{ from (ii)}$$

The equality occurs when $a_1 = a_2, a_3 = a_4, \frac{a_1+a_2}{2} = \frac{a_3+a_4}{2}$ i.e. when $a_1 = a_2 = a_3 = a_4$

Similarly, $a_5a_6a_7a_8 \le \left(\frac{a_5+a_6+a_7+a_8}{4}\right)^4$, the equality occurs when $a_1 = a_2 = ... = a_8$

Proceeding with similar arguments we have $a_1a_2...a_8 \le \left(\frac{a_1+a_2+...+a_8}{8}\right)^8$, the equality occurs when $a_1 = a_2 = ... = a_8$.

Continuing thus, when $n = 2^m$, where m is a positive integer.

$a_1a_2...a_n \le \left(\frac{a_1+a_2+...+a_n}{n}\right)^n$, the equality occurs when $a_1 = a_2 = ... = a_n$.

Let us consider the case when n is not a power of 2.Then $2^{m-1} < n < 2^m$ for some positive integer $m > 1$. In this case there exists a positive integer p such that $n + p = 2^m$. Let us consider $n + p$ positive numbers $a_1, a_2, ..., a_n, a, a, ..., a$ where a is repeated p times and $a = \frac{a_1+a_2+...+a_n}{n}$.

Since $n + p = 2^m$, by what we proved,

$a_1a_2...a_na^p \le \left(\frac{a_1+a_2+...+a_n+pa}{n+p}\right)^{n+p}$, the equality occurs when

$a_1 = a_2 = ... = a_n = a$, i.e. , when $a_1 = a_2 = ... = a_n$.

or, $a_1a_2...a_na^p \le \left(\frac{na+pa}{n+p}\right)^{n+p}$

or, $a_1a_2...a_na^p \le a^{n+p}$,

or, $a_1a_2...a_n \le a^n = \left(\frac{a_1+a_2+...+a_n}{n}\right)^n$

or, $\frac{a_1+a_2+...+a_n}{n} \ge \sqrt[n]{a_1a_2...a_n}$, i.e., $A \ge G$ ……………(iii)

The equality occurs when $a_1 = a_2 = ... = a_n$

Since, $a_1, a_2, ..., a_n$ are positive , $\frac{1}{a_1}, \frac{1}{a_2}, ..., \frac{1}{a_n}$ are all positive.

Now the arithmetic mean of $\frac{1}{a_1}, \frac{1}{a_2}, ..., \frac{1}{a_n}$ is $\frac{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}}{n}$ and the geometric mean of $\frac{1}{a_1}, \frac{1}{a_2}, ..., \frac{1}{a_n}$ is $\left(\frac{1}{a_1}\cdot\frac{1}{a_2}\cdot...\frac{1}{a_n}\right)^{\frac{1}{n}}$. Therefore by previous result ,we have

$\frac{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}}{n} \ge \left(\frac{1}{a_1}\frac{1}{a_2}...\frac{1}{a_n}\right)^{\frac{1}{n}}$, the equality occurs when

$\frac{1}{a_1}=\frac{1}{a_2}=...=\frac{1}{a_n}$ i.e., when $a_1=a_2=...=a_n$

or, $(a_1a_2...a_n)^{\frac{1}{n}} \geq \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+...\frac{1}{a_n}}$

$i,e.$ $G \geq H$, the equality occurs when $a_1=a_2=...=a_n$ ……………….(iv)

Combining (iii) and (iv) we have, $A \geq G \geq H$, the equality occurs when $a_1=a_2=...=a_n$

2nd Part: $A=\frac{a_1+a_2+...+a_n}{n}, G=\sqrt[n]{a_1.a_2...a_n}$

$1+A=\frac{(1+a_1)+(1+a_2)+...+(1+a_n)}{n} \geq \sqrt[n]{(1+a_1)(1+a_2)...(1+a_n)}$

$\Rightarrow (1+A)^n \geq (1+a_1)(1+a_2)...(1+a_n)..................(i)$

Again $(1+a_1)(1+a_2)=1+(a_1+a_2)+a_1a_2$

$\Rightarrow (1+a_1)(1+a_2) \geq 1+2\sqrt{a_1a_2}+a_1a_2=\left(1+\sqrt{a_1a_2}\right)^2$

Similarly $(1+a_1)(1+a_2)(1+a_3)=1+(a_1+a_2+a_3)+(a_1a_2+a_2a_3+a_3a_1)+a_1a_2a_3$

$\Rightarrow (1+a_1)(1+a_2)(1+a_3) \geq 1+3\sqrt{a_1a_2a_3}+3\sqrt{(a_1a_2a_3)^2}+a_1a_2a_3=\left(1+\sqrt[3]{a_1a_2a_3}\right)^3$

Continuing this process upto n times we get

$(1+a_1)(1+a_2)...(1+a_n) \geq \left(1+\sqrt[n]{a_1a_2...a_n}\right)^n$

$\Rightarrow (1+a_1)(1+a_2)...(1+a_n) \geq (1+G)^n..............(ii)$

Combining (i) and (ii) we get $(1+A)^n \geq (1+a_1)(1+a_2)...(1+a_n) \geq (1+G)^n$

Q21. If a,b,c,d be positive real numbers, not all equal , prove that

$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)>16$

☺**:** Let a,b,c,d be positive real numbers, not all equal. Applying A.M.$>$H.M. for a,b,c,d we have, $\frac{a+b+c+d}{4}>\frac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}$

$or, (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)>16.$

Q22: If $n>1$, prove that $(1+2+...+n)\left(1+\frac{1}{2}+...+\frac{1}{n}\right)>n^2$.

☺ : Applying A.M. $>$ H.M. for the unequal positive numbers $1,2,...,n$ we have,

$$\frac{1+2+...+n}{n}>\frac{n}{1+\frac{1}{2}+...+\frac{1}{n}}$$

Or , $(1+2+...+n)\left(1+\frac{1}{2}+...+\frac{1}{n}\right)>n^2$.

Q23. Prove that $\frac{1}{2\sqrt{n+1}}<\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot...\cdot\frac{2n-1}{2n}<\frac{1}{\sqrt{2n+1}}$.

☺. Let $u_n=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot...\cdot\frac{2n-1}{2n}$

Now, $\frac{1}{2}<\frac{2}{3}$, $\frac{3}{4}<\frac{4}{5},\frac{5}{6}<\frac{6}{7},\ldots,\frac{2n-1}{2n}<\frac{2n}{2n+1}$

Combining them,

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot...\cdot\frac{2n-1}{2n}<\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot...\cdot\frac{2n}{2n+1}$$

or, $u_n<\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot...\cdot\frac{2n}{2n+1}$

Therefore, $u_n^{\,2}=u_n.u_n<\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot...\cdot\frac{2n-1}{2n}\right)\left(\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot...\cdot\frac{2n}{2n+1}\right)=\frac{1}{2n+1}$

i.e., $u_n<\frac{1}{\sqrt{2n+1}}$ ……(i)

Also, $(2n+1)u_n=\frac{1}{2}\frac{3}{4}\frac{5}{6}...\frac{2n-1}{2n}(2n+1)$

$$=\frac{3}{2}\frac{5}{4}\frac{7}{6}...\frac{2n+1}{2n}$$

Now, $\frac{3}{2}>\frac{4}{3},\frac{5}{4}>\frac{6}{5},\frac{7}{6}>\frac{8}{7},\ldots,\frac{2n+1}{2n}>\frac{2n+2}{2n+1}$.

Combining them,

$$\frac{3}{2}\frac{5}{4}\frac{7}{6}...\frac{2n+1}{2n}>\frac{4}{3}\frac{6}{5}\frac{8}{7}...\frac{2n+2}{2n+1}.$$

$$\therefore (2n+1)^2u_n^{\,2}=(2n+1)u_n(2n+1)u_n>\left(\frac{3}{2}\frac{5}{4}\frac{7}{6}...\frac{2n+1}{2n}\right)\left(\frac{4}{3}\frac{6}{5}\frac{8}{7}...\frac{2n+2}{2n+1}\right)=n+1$$

Or, $(2n+1)u_n>\sqrt{n+1}$

Or, $u_n>\frac{\sqrt{n+1}}{2n+1}>\frac{\sqrt{n+1}}{2n+2}=\frac{1}{2\sqrt{n+1}}$ ……(ii)

From (i) and (ii) it follows that $\frac{1}{2\sqrt{n+1}}<\frac{1}{2}\frac{3}{4}\frac{5}{6}...\frac{2n-1}{2n}<\frac{1}{\sqrt{2n+1}}$

Q24. Show that $(a^3+b^3+c^3)\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)>9$.

☺. Applying A.M.>H.M. for the unequal positive numbers a^3,b^3,c^3 we have,

$\frac{a^3+b^3+c^3}{3}>\frac{3}{\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}}$ or, $(a^3+b^3+c^3)\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)>9$.

Q25. Show that $a^4+b^4+c^4 \ge abc\left(a+b+c\right)$.

☺**.** Applying mth power theorem with $m=4$ for the positive real numbers a,b,c we have,

$\dfrac{a^4+b^4+c^4}{3} \ge \left(\dfrac{a+b+c}{3}\right)^4$, the equality occurs when $a=b=c$.

Or, $a^4+b^4+c^4 \ge \left(\dfrac{a+b+c}{3}\right)^3 \left(a+b+c\right) \ge abc\left(a+b+c\right)$,[Applying A.M. $\ge$G.M. for a,b,c] , the equality occurs when $a=b=c$.

Q26. Show that $\left(\dfrac{a+b+c}{3}\right)^3 \ge a\left(\dfrac{b+c}{2}\right)^2$.

Q27. Show that $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a} \ge 3$.

☺**. Hints:** Apply A.M.$\ge$G.M. for the positive real numbers $\dfrac{a}{b},\dfrac{b}{c},\dfrac{c}{a}$.

Q28. Show that $\left(a^2b+b^2c+c^2a\right)\left(ab^2+bc^2+ca^2\right) \ge 9a^2b^2c^2$.

☺**. Hints:** Apply A.M.$\ge$G.M. for the positive real numbers a^2b,b^2c,c^2a and ab^2,bc^2,ca^2.

Q29. Show that $\dfrac{9}{a+b+c} \le \dfrac{2}{a+b}+\dfrac{2}{b+c}+\dfrac{2}{c+a} \le \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$.

☺**.** Applying A.M.$\ge$H.M. for the positive numbers $\dfrac{2}{a+b},\dfrac{2}{b+c},\dfrac{2}{c+a}$ we have,

$$\frac{\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}}{3} \ge \frac{3}{\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}} = \frac{3}{a+b+c}$$

, the eqlality occurs when $\dfrac{2}{a+b}=\dfrac{2}{b+c}=\dfrac{2}{c+a}$ i.e. when $a=b=c$.

Or, $\dfrac{9}{a+b+c} \le \dfrac{2}{a+b}+\dfrac{2}{b+c}+\dfrac{2}{c+a}$, the equality occurs when $a=b=c$. …(i)

Applying A.M.$\ge$H.M. for the positive numbers a,b we have,

$\dfrac{a+b}{2} \ge \dfrac{2}{\frac{1}{a}+\frac{1}{b}}$, the equality occurs when $a=b$.

Or, $\dfrac{2}{a+b} \le \dfrac{1}{2}\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$, the equality occurs when $a=b$.

Similarly, $\dfrac{2}{b+c} \le \dfrac{1}{2}\left(\dfrac{1}{b}+\dfrac{1}{c}\right)$ and $\dfrac{2}{c+a} \le \dfrac{1}{2}\left(\dfrac{1}{c}+\dfrac{1}{a}\right)$, the equality occurs when $b=c$ and $c=a$.

Adding , $\dfrac{2}{a+b}+\dfrac{2}{b+c}+\dfrac{2}{c+a} \le \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$, the equality occurs when $a=b=c$… (ii).

From (i) and (ii) we have,

$\frac{9}{a+b+c} \le \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \le \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, the equality occurs when $a=b=c$.

Q29. If x,y,z are positive real numbers and $x+y+z=1$, prove that

$$8xyz \le (1-x)(1-y)(1-z) \le \frac{8}{27}$$

☺ **.** $1-x=y+z,\quad 1-y=z+x,\ 1-z=x+y.$

We have $\frac{y+z}{2} \ge \sqrt{yz},\quad \frac{z+x}{2} \ge \sqrt{zx},\quad \frac{x+y}{2} \ge \sqrt{xy}.$

Therefore $(y+z)(z+x)(x+y) \ge 8xyz$, the equality occurs when $x=y=z$

or, $(1-x)(1-y)(1-z) \ge 8xyz$

Again let us consider three positive real numbers $1-x, 1-y, 1-z$.

Applying $A.M. \ge G.M.$, we have

$\frac{(1-x)+(1-y)+(1-z)}{3} \ge \sqrt[3]{(1-x)(1-y)(1-z)}$, the equality occurs when $1-x=1-y=1-z$ i.e, when $x=y=z$.

or, $\left(\frac{3-1}{3}\right)^3 \ge (1-x)(1-y)(1-z)$

or, $\frac{8}{27} \ge (1-x)(1-y)(1-z)$

or, $8xyz \le (1-x)(1-y)(1-z) \le \frac{8}{27}$, the equality occurs when $x=y=z=\frac{1}{3}$.

Q30. If $x,y,z>0$ and $x+y+z=1$, show that the maximum value of $(1-x)(1-y)(1-z)$ is $\frac{8}{27}$.

Q31. If a,b,c be positive real numbers such that the sum of any two is greater than the third, prove that $abc \ge (a+b-c)(b+c-a)(c+a-b)$.

☺ **.** Applying A.M.$\ge$G.M. for two positive numbers $a+b-c, b+c-a$ we have,

$$\frac{(a+b-c)+(b+c-a)}{2} \ge \sqrt{(a+b-c)(b+c-a)}$$

Or, $b \ge \sqrt{(a+b-c)(b+c-a)}$

Similarly, $c \ge \sqrt{(b+c-a)(c+a-b)}$ and $a \ge \sqrt{(c+a-b)(a+b-c)}$.

Therefore $abc \ge (a+b-c)(b+c-a)(c+a-b)$, the equality occurs when $b+c-a=c+a-b=a+b-c$, i.e., when $a=b=c$.

Q32. If n be positive integer, prove that $\frac{1}{\sqrt{4n+1}} < \frac{3.7.11...(4n-1)}{5.9.13...(4n+1)} < \sqrt{\frac{3}{4n+3}}$.

☺ **.** If r be a positive integer, we have

$$\frac{(4r+1)+(4r-3)}{2} > \sqrt{(4r+1)(4r-3)}$$

or, $4r-1 > \sqrt{(4r+1)(4r-3)}$

or, $\frac{4r-1}{4r+1} > \sqrt{\frac{4r-3}{4r+1}}$.

Taking $r = 1, 2, 3, \ldots, n$ we have

$\frac{3}{5} > \sqrt{\frac{1}{5}}$, $\frac{7}{9} > \sqrt{\frac{5}{9}}$, $\frac{11}{13} > \sqrt{\frac{9}{13}}, \ldots, \frac{4n-1}{4n+1} > \sqrt{\frac{4n-3}{4n+1}}$.

Therefore, $\frac{3.7.11\ldots(4n-1)}{5.9.13\ldots(4n+1)} > \frac{1}{\sqrt{4n+1}}$.

Again , if r be a positive integer, we have

$$\frac{(4r-1)+(4r+3)}{2} > \sqrt{(4r-1)(4r+3)}$$

or, $(4r+1) > \sqrt{(4r-1)(4r+3)}$

or, $\frac{4r+1}{4r-1} > \sqrt{\frac{4r+3}{4r-1}}$.

Taking $r = 1, 2, 3, \ldots, n.$ we have,

$\frac{5}{3} > \sqrt{\frac{7}{3}}, \frac{9}{7} > \sqrt{\frac{11}{7}}, \frac{13}{11} > \sqrt{\frac{15}{11}}, \ldots, \frac{4n+1}{4n-1} > \sqrt{\frac{4n+3}{4n-1}}$.

Therefore, $\frac{5.9.13\ldots(4n+1)}{3.7.11\ldots(4n-1)} > \sqrt{\frac{4n+3}{3}}$

or, $\frac{3.7.11\ldots(4n-1)}{5.9.13\ldots(4n+1)} < \sqrt{\frac{3}{4n+3}}$.

Therefore , $\frac{1}{\sqrt{4n+1}} < \frac{3.7.11\ldots 4n-1}{5.9.13\ldots 4n+1} < \sqrt{\frac{3}{4n+3}}$

Q33. If a, b, c be positive real numbers and $abc = k^3$, prove that

$$(1+a)(1+b)(1+c) \geq (1+k)^3.$$

☺ **.**We have $(1+a)(1+b)(1+c) = 1 + \sum a + \sum ab + abc$

Now, Applying $\text{A.M.} \geq \text{G.M.}$ for the positive numbers a, b, c and ab, bc, ca

$\frac{\sum a}{3} \geq \sqrt[3]{abc}$ and $\frac{\sum ab}{3} \geq \sqrt[3]{a^2b^2c^2}$

i.e, $\sum a \geq 3k$ and $\sum ab \geq 3k^2$.

Therefore, $(1+a)(1+b)(1+c) \geq 1 + 3k + 3k^2 + k^3$

i.e, $(1+a)(1+b)(1+c) \geq (1+k)^3$

Q34. If a, b, c, d be all positive and $abcd = k^4$, then prove that

$$(1+a)(1+b)(1+c)(1+d) \geq (1+k)^4.$$

Q35. If $a_1, a_2, ..., a_n$ be all positive real numbers and $a_1 a_2 ....a_n = k^n$ $(k>0)$, prove that $(1+a_1)(1+a_2)...(1+a_n) \geq (1+k)^n$.

Q36. If $x_i > a > 0$ for $i = 1, 2, ..., n$ and $(x_1 - a)(x_2 - a)...(x_n - a) = k^n$ $(k > 0)$, prove that $x_1 x_2 ... x_n \geq (a+k)^n$.

☺. Let $y_i = x_i - a$ for $i = 1, 2, ..., n$. Then $y_i > 0$ for $i = 1, 2, 3, ..., n$ and $y_1 y_2 ... y_n = k^n$.

Now, $x_1 x_2 ... x_n = (a + y_1)(a + y_2)...(a + y_n) = a^n + a^{n-1} \sum y_1 + a^{n-2} \sum y_1 y_2$
$+ ... + a \sum y_1 y_2 ... y_{n-1} + y_1 y_2 ... y_n$ ………………(1)

Applying A.M.≥G.M. for the n positive numbers $y_1, y_2, ..., y_n$ we have
$\frac{\sum y_1}{n} \geq (y_1 y_2 ... y_n)^{\frac{1}{n}}$ i.e., $\sum y_1 \geq nk$.

Again applying $A.M. \geq G.M.$ for the ${}^n c_2$ positive numbers $y_1 y_2, y_1 y_3, ...$ we have,
$\frac{\sum y_1 y_2}{{}^n c_2} \geq \left\{ (y_1 y_2 ... y_n)^{n-1} \right\}^{\frac{1}{{}^n c_2}}$ i.e., $\sum y_1 y_2 \geq {}^n c_2 \left(k^{n(n-1)} \right)^{\frac{2}{n(n-1)}} = {}^n c_2\ k^2$

Similarly, $\sum y_1 y_2 y_3 \geq {}^n c_3 k^3, \ldots, \sum y_1 y_2 ... y_{n-1} \geq {}^n c_{n-1} k^{n-1}$.

Therefore, from (1) we have,
$x_1 x_2 ... x_n \geq a^n + {}^n c_1 a^{n-1} k + {}^n c_2 a^{n-2} k^2 + ... + {}^n c_{n-1} a k^{n-1} + k^n = (a+k)^n$.

Q37. If a, b, c are positive real numbers and $abc = 1$ then prove that the least value of $(1+a)(1+b)(1+c)$ is 8.

Q38. If $a_1, a_2, ..., a_5$ be positive real numbers, prove that

$$\left(\frac{a_1 + a_2 + ... + a_5}{5} \right)^5 \geq \left(\frac{a_1 + a_2}{2} \right)^2 \left(\frac{a_3 + a_4 + a_5}{3} \right)^3$$

☺.**Hints:** Consider two positive numbers $\frac{a_1 + a_2}{2}, \frac{a_3 + a_4 + a_5}{3}$ with associated weights 2 and 3 respectively.

Q39. If $a_1, a_2, ..., a_n$ be n positive numbers and $a_n a_{n-1} = 1$, then show that

$$\left(\frac{a_1 + a_2 + ... + a_n}{n} \right)^n \geq \left(\frac{a_1 + a_2 + ... + a_{n-2}}{n-2} \right)^{n-2}.$$

☺.Let us consider the set a, a_{n-1}, a_n where $a = \frac{a_1 + a_2 + ... + a_{n-2}}{n-2}$ with associated weights $(n-2)$,1,1 respectively.

Applying $A.M. \geq G.M.$ we get, $\frac{(n-2).a + 1.a_{n-1} + 1.a_n}{(n-2)+1+1} \geq \left(a^{n-2} a_{n-1} a_n \right)^{\frac{1}{n}}$

or, $\left(\frac{a_1+a_2+...+a_n}{n}\right)^n \geq \left(\frac{a_1+a_2+...+a_{n-2}}{n-2}\right)^{n-2}$, since $a_n a_{n-1}=1$.

Q40. If $a>0$ but $\neq 1$ and x,y,z are rational numbers decending order of magnitude, prove that $a^x(y-z)+a^y(z-x)+a^z(x-y)>0$.

☺**.** Since $x>y>z$, we have $x-y>0$ and $y-z>0$.

Let us consider two unequal positive numbers a^x, a^z with associated positive rational weights $y-z$ and $x-y$ respectively. Then we have

$$\frac{(y-z)a^x+(x-y)a^z}{x-z} > \left[a^{x(y-z)+z(x-y)}\right]^{\frac{1}{x-z}}$$

Or, $(y-z)a^x+(x-y)a^z > (x-z)a^y$

Or, $a^x(y-z)+a^y(z-x)+a^z(x-y)>0$.

Q41. If a be a positive real number, not equal to 1, and x,y are positive rational numbers, then $\frac{a^x-1}{x} > \frac{a^y-1}{y}$ if $x>y$.

☺**.** Since $a>0$, and $\neq 1$, $a^x>0$ and $\neq 1$.

Let us consider two unequal positive numbers a^x and 1 with associated positive rational weights y and $x-y$ respectively.

Then, $\frac{y.a^x+(x-y).1}{x} > \left[a^{xy}.1\right]^{\frac{1}{x}}$

Or, $y.a^x+x-y > xa^y$

Or, $y(a^x-1) > x(a^y-1)$

Or, $\frac{a^x-1}{x} > \frac{a^y-1}{y}$, since $x>0, y>0$.

Theorem: If a be a positive real number, not equal to 1, and m be a rational numbers, then $a^m-1 > \text{or} < m(a-1)$, according as m does not or does lie between 0 and 1.

Theorem: If $a_1,a_2,...,a_n$ be n positive numbers, not all equal, and m be a rational number, then $\frac{a_1{}^m+a_2{}^m+...+a_n{}^m}{n} >$ or $< \left(\frac{a_1+a_2+...+a_n}{n}\right)^m$, according as m does not or does lie between 0 and 1.

Proof: Let $k=\frac{a_1+a_2+...+a_n}{n}$. Then $\frac{a_1}{k},\frac{a_2}{k},...,\frac{a_n}{k}$ are all positive and not all of them are equal to 1.

Let $1\leq i\leq n$. Then $\left(\frac{a_i}{k}\right)^m-1>$ or $<m\left(\frac{a_i}{k}-1\right)$ according as m does not or does lie between 0 and 1. The inequality reduce to an equality if $\frac{a_i}{k}=1$.

Considering n such relations for $i=1,2,3,...,n$ and noting that all of them are not equalities , we have

$\left(\frac{a_1}{k}\right)^m+\left(\frac{a_2}{k}\right)^m+...+\left(\frac{a_n}{k}\right)^m-n>$ or $<m\left(\frac{a_1}{k}+\frac{a_2}{k}+...+\frac{a_n}{k}-n\right)$, according as m does not or does lie between 0 and 1.

Or, $\frac{a_1{}^m+a_2{}^m+...+a_n{}^m}{k^m}-n>$ or $<m(n-n)$

Or, $\frac{a_1{}^m+a_2{}^m+...+a_n{}^m}{k^m}>$ or $<n$

Or, $\frac{a_1{}^m+a_2{}^m+...+a_n{}^m}{n}>$ or $<\left(\frac{a_1+a_2+...+a_n}{n}\right)^m$, according as m does not or does lie between 0 and 1.

Q42. If n be a positive integer, prove that $\left(1+\frac{1}{n+1}\right)^{n+1}>\left(1+\frac{1}{n}\right)^n$.

☺**.** Let us consider $n+1$ positive numbers $1+\frac{1}{n},1+\frac{1}{n},...,1+\frac{1}{n}$ (n times) and 1 .

Applying A.M.$>$G.M. we have, $\frac{n\left(1+\frac{1}{n}\right)+1}{n+1}>\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$

Or, $\left(\frac{n+1+1}{n+1}\right)^{n+1}>\left(1+\frac{1}{n}\right)^n$

Or, $\left(1+\frac{1}{n+1}\right)^{n+1}>\left(1+\frac{1}{n}\right)^n$

Q43. If a,b,c be unequal positive real numbers such that the sum of any two is greater than the third , prove that $\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}>\frac{9}{a+b+c}$.

☺**.Hints:** Applying A.M.>H.M. for unequal numbers $\frac{1}{b+c-a},\frac{1}{c+a-b},\frac{1}{a+b-c}$.

Q44. If a,b,c are positive real numbers and $a+b=4$, prove that

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\geq\frac{25}{2}.$$

☺**.** $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2=(a^2+b^2)+(a^{-2}+b^{-2})+4$.

We have $\frac{a^m+b^m}{2}>$ or $<\left(\frac{a+b}{2}\right)^m$, according as m does not or does lie between 0 and 1.

Let $m=2$. Then $\frac{a^2+b^2}{2}\geq\left(\frac{a+b}{2}\right)^2$, the equality occurs when $a=b$.

Or, $a^2+b^2\geq 8$.

Let $m=-2$. Then $\frac{a^{-2}+b^{-2}}{2}\geq\left(\frac{a+b}{2}\right)^{-2}$, the equality occurs when $a=b$.

Or, $a^{-2}+b^{-2}\geq\frac{1}{2}$.

Therefore $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\geq 4+8+\frac{1}{2}=\frac{25}{2}$.

Q45. If a,b,c be positive and $a+b+c=1$ prove that

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2\geq 33\frac{1}{3}.$$

Q46. If a,b,c be three positive real numbers in harmonic progression and n be a positive integer greater than 1, prove that $a^n+c^n>2b^n$.

☺**.** Since a,c are unequal positive real numbers and $n>1$,

$$\frac{a^n+c^n}{2}>\left(\frac{a+c}{2}\right)^n \text{.........(i)}$$

Since a,b,c are in H.P., $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$.

But $\frac{a^{-1}+c^{-1}}{2}>\left(\frac{a+c}{2}\right)^{-1}$, since a,c are unequal positive real numbers.

Therefore, $\frac{1}{b}>\left(\frac{a+c}{2}\right)^{-1}$, i.e., $\left(\frac{a+c}{2}\right)>b$.

Therefore from (i), $a^n+c^n>2b^n$.

Q47. If x,y,z are positive real numbers such that $x^2+y^2+z^2=27$ show that $x^3+y^3+z^3\geq 81$.

☺**.Hints:** Apply mth ($m=\frac{3}{2}$)power theorem for the numbers x^2,y^2,z^2

Q48. If a,b,c be positive and $a+b+c=1$, then show that

$$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)\geq 8.$$

Q49. If a,b,c,d be four positive real numbers and $a+b+c+d=s$ then show that

$(i)\ 81abcd\leq(s-a)(s-b)(s-c)(s-d)\leq\frac{81}{256}s^4$

$(ii)\ \frac{16}{s}\leq\frac{3}{s-a}+\frac{3}{s-b}+\frac{3}{s-c}+\frac{3}{s-d}\leq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$

☺. (i) Let us consider four positive numbers $(s-a),(s-b),(s-c),(s-d)$ and apply $A.M. \geq G.M.$ we have,

$$\frac{s-a+s-b+s-c+s-d}{4} \geq \{(s-a)(s-b)(s-c)(s-d)\}^{\frac{1}{4}}.$$

Or, $\left(\frac{3s}{4}\right)^4 \geq (s-a)(s-b)(s-c)(s-d)$.

Or, $(s-a)(s-b)(s-c)(s-d) \leq \frac{81}{256}s^4$ ………………..(1)

Now, applying $A.M. \geq G.M.$ for three positive numbers b,c,d. We have,

$$\frac{b+c+d}{3} \geq (bcd)^{\frac{1}{3}}$$

$\therefore s-a \geq 3(bcd)^{\frac{1}{3}}$ …………..(2)

Similarly, $s-b \geq 3(acd)^{\frac{1}{3}}$ …………(3)

$s-c \geq 3(abd)^{\frac{1}{3}}$ …………………(4)

$s-d \geq 3(abc)^{\frac{1}{3}}$ ………………….(5)

Multiplying, (2),(3), (4), and (5) we get,

$(s-a)(s-b)(s-c)(s-d) \geq 81\,(bcd.acd.abd.abc)^{\frac{1}{3}}$.

$\therefore$ $(s-a)(s-b)(s-c)(s-d) \geq 81abcd$ …………….(6)

Combining (1) and (6) we get,

$81abcd \leq (s-a)(s-b)(s-c)(s-d) \leq \frac{81}{256}s^4$.

(ii) Clearly $s-a>0, s-b>0, s-c>0, s-d>0$

Applying $A.M \geq G.M$ on the four numbers $s-a>0, s-b>0, s-c>0, s-d>0$ we get

$$\frac{(s-a)+(s-b)+(s-c)+(s-d)}{4} \geq \frac{4}{\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}+\frac{1}{s-d}}$$

$$\Rightarrow \frac{3s}{4} \geq \frac{4}{\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}+\frac{1}{s-d}}$$

$$\Rightarrow \frac{3}{s-a}+\frac{3}{s-b}+\frac{3}{s-c}+\frac{3}{s-d} \geq \frac{16}{s}$$

$$\Rightarrow \frac{16}{s} \leq \frac{3}{s-a}+\frac{3}{s-b}+\frac{3}{s-c}+\frac{3}{s-d}$$

Again applying $H.M. \le A.M.$ on three non negative real numbers a,b,c we get

$$\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \le \frac{a+b+c}{3} \Rightarrow \frac{9}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \le (s-d) \Rightarrow \frac{9}{s-d} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Similarly we can write

$$\frac{9}{s-b} \le \frac{1}{a}+\frac{1}{c}+\frac{1}{d};\frac{9}{s-c} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{d};\frac{9}{s-d} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c};$$

Adding we get $\frac{3}{s-a}+\frac{3}{s-b}+\frac{3}{s-c}+\frac{3}{s-d} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$

Combining we get $\frac{16}{s} \le \frac{3}{s-a}+\frac{3}{s-b}+\frac{3}{s-c}+\frac{3}{s-d} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$

.