Kabi Jagadram Roy Govt. General Degree College

Department of Mathematics

Analytic Geometry

Lecture Note by Dr. Chinmay Das

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BKU Syllabus

Reflection properties of conics, Transformation of axes and second degree equations, Invariant, classification of conics using the discriminant, Pair of straight lines, polar equations of straight lines,
Spheres, Cone, Cylindrical surfaces. Central conicoids, paraboloids, plane sections of conicoids, Tangent, Normal, Enveloping Cone and Cylinder, Generating lines, classification of quadratics, Transformation of axes in space and general equation of second degree. circles and conics.

Content

Chapter 1

Transformation of Axes

Transformation of axes in geometry refers to the process of changing the coordinate system used to represent points and objects in space. This transformation is a fundamental concept in geometry and is often used to simplify calculations, study symmetries, and analyze geometric figures from different perspectives. There are two primary types of transformations of axes: translation and rotation.

1.1 Translation of Axes

Translation involves shifting the entire coordinate system in a specified direction. This is typically done to simplify calculations or to align the axes with specific features of a geometric figure.

In a two-dimensional space, you can translate the axes by adding or subtracting constant values from the x and y coordinates of every point. For example, if you want to shift the origin from (0,0) to (α, β), you add α to the x-coordinates and β to the y-coordinates of all points.

In three-dimensional space, a similar process is applied to all three axes.

Translation does not change the size, shape, or orientation of geometric objects; it only changes their position within the coordinate system.

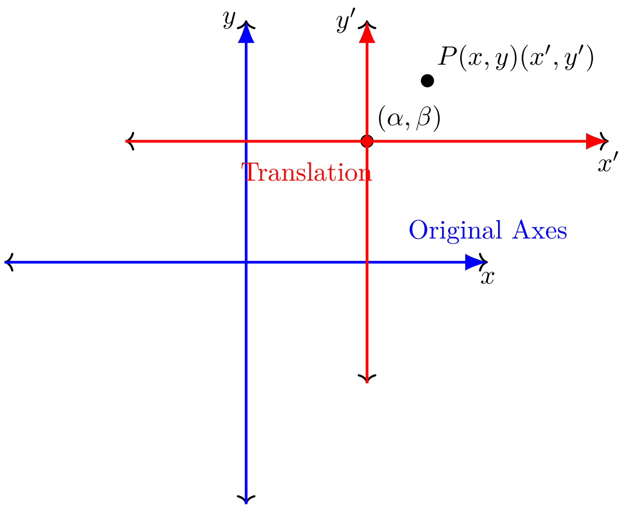

Figure 1.1: Translation of Axes

Translation:

$$\begin{matrix} x = x' + \alpha \\ y = y' + \beta \end{matrix} \Rightarrow \mathbf{X} = \mathbf{X}' + \mathbf{b} \text{ where } \mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{X}' = \begin{pmatrix} x' \\ y' \end{pmatrix} \ \& \ \mathbf{b} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Note 1:

In the equation of a locus referred to a original system of axes (x, y) will be replaced by $(x' + \alpha, y' + \beta)$ when the equation is referred to new pair of axes. Inversely (x', y') will be replaced by $(x - \alpha, y - \beta)$.

1.2 Rotation of Axes

Rotation involves changing the orientation of the coordinate axes while keeping the origin fixed. This transformation is often used to analyze the symmetry and properties of geometric figures.

In two dimensions, you can rotate the axes by a specified angleθ. This results in a new set of axes, often denoted as x' and y', that are rotated by θ degrees from the original axes.

In three-dimensional space, you can perform similar rotations about the x, y and z axes.

Rotation preserves the size and shape of geometric objects but may alter their orientation.

Transformation of axes is a valuable tool in various fields, including physics, engineering, computer graphics, and mathematics. It simplifies problem-solving and allows for a more intuitive analysis of geometric problems by aligning coordinate systems with the inherent symmetries or characteristics of the objects being studied.

To change the directions of the axes without changing the origin. Let OX and OY be the original axes and OX', OY' the new axes which are obtained by rotating the original axes through an angle θ. Rotation is taken in anti-clockwise direction.

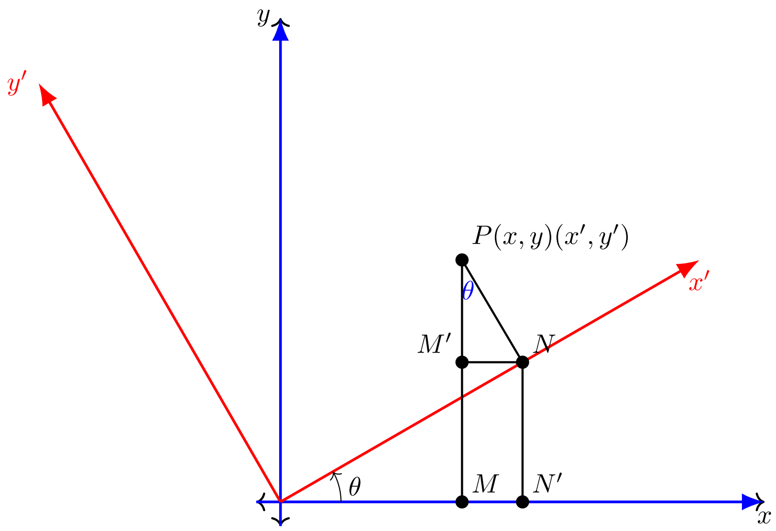

Figure 1.2: Translation of Axes

Let (x, y) be the coordinates of a point P with respect to original axes and (x', y') the coordinates of P with respect to new axes. Draw PM and PN perpendiculars to OX and OX' and also NN' and NM' perpendiculars to OX and PM, respectively. Then

$$OM = x, MP = y, ON = x', NP = y'$$

and $\angle M'PN = \theta$. Now, we have

$$x = OM = ON' - MN' = ON' - M'N = ON\cos\theta - NP\sin\theta$$
$$= x'\cos\theta - y'\sin\theta$$

and

$$y = MP = MM' + M'P = N'N + M'P = ON\sin\theta + NP\cos\theta$$
$$= x'\sin\theta + y'\cos\theta$$

Hence the relations connecting the two systems of axes are given by

$$\left.\begin{array}{l} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{array}\right\} \tag{1.1}$$

Note 2:

An equivalent form of the relations (1.1) is

$$\left.\begin{array}{l} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \end{array}\right\}.$$

Rotation:

$$\begin{array}{l} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{array} \Rightarrow \mathbf{X} = \mathbf{A}\mathbf{X}' \text{ where } \mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{X}' = \begin{pmatrix} x' \\ y' \end{pmatrix} \ \& \ \mathbf{A} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

Transformation: Translation & Rotation

$$\begin{array}{l} x = x'\cos\theta - y'\sin\theta + \alpha \\ y = x'\sin\theta + y'\cos\theta + \beta \end{array} \Rightarrow \mathbf{X} = \mathbf{A}\mathbf{X}' + \mathbf{b} \text{ where}$$

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{X}' = \begin{pmatrix} x' \\ y' \end{pmatrix}, \mathbf{A} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \ \& \ \mathbf{b} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

since $det(\mathbf{A}) = 1$ (i.e. matrix $\mathbf{A}$ is orthogonal) this transformation is called an orthogonal transformation.

1.3 General Orthogonal Transformation

Let (x, y) be the coordinates of a point P referred to rectangular axes OX and OY and (x', y') be the coordinates of the same point referred to a new set of rectangular axes. O′X′ and O′Y′ whose equations are $lx + my + n = 0$ and $mx - ly + k = 0$ with respect to OX and OY: Perpendicular distances from P to $mx - ny +\ k = 0$ is

$$N'P = x' = \pm\frac{mx - ly + k}{\sqrt{l^2 + m^2}} \tag{1.2}$$

Perpendicular distance from P to $lx + my + n = 0$ is

$$M'P = y' = \pm\frac{lx + my + n}{\sqrt{l^2 + m^2}}. \tag{1.3}$$

The same sign is taken in both cases according to convenience. By (1.2) and (1.3) values of x and y are found out in terms of x' and y'.

1.4 Invariants in orthogonal transformation

. Relations connecting the coefficients of an expression or some other quantity which remain unchanged under an orthogonal transformation are called invariants under that orthogonal transformation.

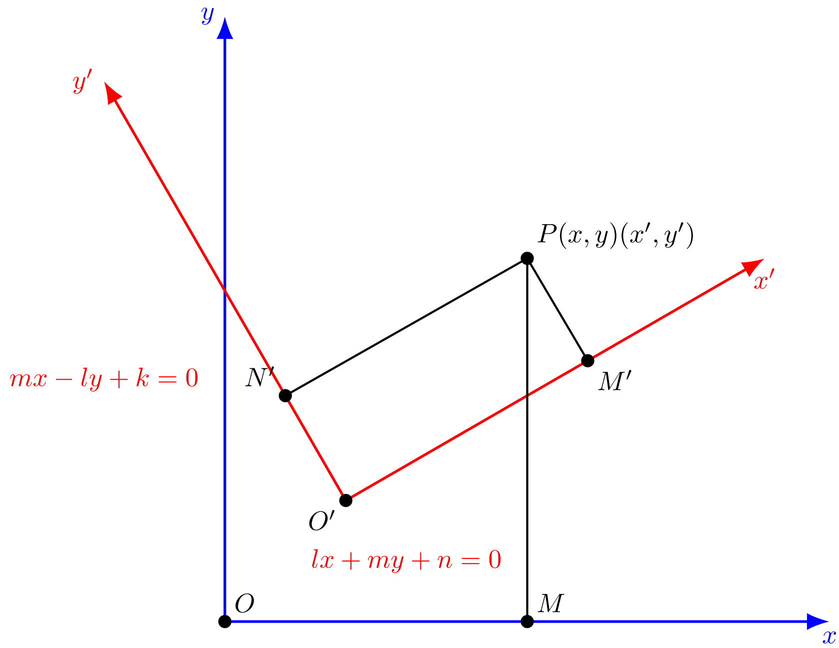

Figure 1.3: Translation of Axes

Invariants

If, by the orthogonal transformation without change of origin, the expression $\left(ax^2+2hxy+by^2+2gx+2fy+c\right)$ be changed into $\left(a'X^2+2h'XY+b'Y^2+2g'X+2f'Y+c'\right)$, then

1. $a'+b'=a+b$;
2. $a'b'-h'^2=ab-h^2$;
3. $g'^2+f'^2=g^2+f^2$;
4. $$\begin{aligned}&b'c'+c'a'+a^{\aleph}b'-f'^2-g'^2-h'^2\\&=bc+ca+ab-f^2-g^2-h^2.\end{aligned}$$

By applying the formulæ for orthogonal transformation, viz.

$$x = X\cos\theta - Y\sin\theta, y = X\sin\theta + Y\cos\theta,$$

the given expression changes to

$$\begin{aligned}
a(X\cos\theta-&Y\sin\theta)^2 + 2h(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)\\
&+ b(X\sin\theta + Y\cos\theta)^2 + 2g(X\cos\theta - Y\sin\theta)\\
&+ 2f(X\sin\theta + Y\cos\theta) + c\\
=&X^2\left(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta\right)\\
&+ 2XY\left\{h\left(\cos^2\theta - \sin^2\theta\right) - (a-b)\sin\theta\cos\theta\right\}\\
&+ Y^2\left(a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta\right)\\
&+ 2X(g\cos\theta + f\sin\theta) + 2Y(f\cos\theta - g\sin\theta) + c
\end{aligned}$$

$$\begin{aligned}
\text{so that}\quad &\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta,\\
a' =& h\left(\cos^2\theta - \sin^2\theta\right) - (a-b)\sin\theta\cos\theta,\\
h' =& a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta,\\
\boldsymbol{g}' =& g\cos\theta + f\sin\theta, f' = f\cos\theta - g\sin\theta, c' = c.
\end{aligned}$$

So that

$$\begin{aligned}
a' &= a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta \\
h' &= h\left(\cos^2\theta - \sin^2\theta\right) - (a-b)\sin\theta\cos\theta \\
b' &= a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta \\
g' &= g\cos\theta + f\sin\theta, f' = f\cos\theta - g\sin\theta, c' = c.
\end{aligned}$$

Hence we have $a' + b' = a + b$, since $\sin^2\theta + \cos^2\theta = 1$.
Again $2a' = a(1+\cos 2\theta) + 2h\sin 2\theta + b(1-\cos 2\theta)$

$$= (a+b) + \{2h\sin 2\theta + (a-b)\cos 2\theta\}.$$

Similarly, $2b' = (a+b) - \{2h\sin 2\theta + (a-b)\cos 2\theta\}$.
Therefore $4a'b' = (a+b)^2 - \{2h\sin 2\theta + (a-b)\cos 2\theta\}^2$. Also $4h'^2 = \{2h\cos 2\theta - (a-b)\sin 2\theta\}^2$.
Therefore $4\left(a'b' - h'^2\right) = (a+b)^2 - \left\{4h^2 + (a-b)^2\right\}$

$$= 4\left(ab - h^2\right).$$

Hence $a'b' - h'^2 = 'ab - h^2$.
Again

$$\begin{aligned}
g'^2 + f'^2 &= (g\cos\theta + f\sin\theta)^2 + (f\cos\theta - g\sin\theta)^2 \\
&= g^2 + f^2.
\end{aligned}$$

Lastly $b'c' + c'a' + a'b' - f'^2 - g'^2 - h'^2$

$$\begin{aligned}
&= c'\left(a' + b'\right) + \left(a'b' - h'^2\right) - \left(f'^2 + g'^2\right) \\
&= c(a+b) + \left(ab - h^2\right) - \left(f^2 + g^2\right) \\
&= bc + ca + ab - f^2 - g^2 - h^2.
\end{aligned}$$

1.5 Illustrative Examples:

Example 1.5.1

Find the equation of the line $\frac{x}{a} + \frac{y}{b} = 1$ when the origin is shifted to the point (a, b).

Solution: Since the origin is shifted to the point (a, b), then $x = X + a, y = Y + b$.
The given equation is $\frac{x}{a} + \frac{y}{b} = 1$.
Then

$$\frac{X+a}{a} + \frac{Y+b}{b} = 1 \Rightarrow \frac{X}{a} + 1 + \frac{Y}{b} + 1 = 1 \Rightarrow \frac{X}{a} + \frac{Y}{b} = -1.$$

This is the transformed form of the given line.

Example 1.5.2

Transform to axes inclined at 30° to the original axes

$$x^2 + 2\sqrt{3}xy - y^2 - 2 = 0$$

Solution: The transformation formulæ are

$$\begin{aligned}
x &= x'\cos 30^\circ - y'\sin 30^\circ = \frac{1}{2}\left(x'\sqrt{3} - y'\right), \\
y &= x'\sin 30^\circ + y'\cos 30^\circ = \frac{1}{2}\left(x' + y'\sqrt{3}\right).
\end{aligned}$$

The transformed equation is

$$\left(x'\sqrt{3} - y'\right)^2 + 2\sqrt{3}\left(x'\sqrt{3} - y'\right)\left(x' + y'\sqrt{3}\right) - \left(x' + y'\sqrt{3}\right)^2 = 8$$

or, $x^2 - y^2 = 1$.

Example 1.5.3

Transform the equation $y^2 - 2y = x$ w. r. t. parallel axes through $(-1, 1)$.

Solution: Since the origin is shifted to the point $(-1,1)$, then $x = X-1, y = Y+1$. The given equation is $y^2 - 2y = x$. So;

$$\begin{aligned}
&(Y+1)^2 - 2(Y+1) = X-1\\
&\Rightarrow (Y+1)(Y-1) = X-1\\
&\Rightarrow Y^2 - 1 = X - 1 \Rightarrow Y^2 = X
\end{aligned}$$

This is the transformed form of the given equation.

Example 1.5.4

By shifting the origin to the point (α, β) without changing the directions of axes, each of the equations $x - y + 3 = 0$ and $2x - y + 1 = 0$ is reduced to the form $ax' + by' = 0$. Find α, β.

Solution: Since the origin is shifted to the point (α, β), then

$$x = x' + \alpha, \quad y = y' + \beta.$$

The given equations are $x + y + 3 = 0$ and $2x - y + 1 = 0$. Then

$$2\,(x' + \alpha) - (y' + \beta) + 1 = 0 \text{ and } (x' + \alpha) - (y' + \beta) + 3 = 0.$$

Since given equations takes the from $ax' + by' = 0$, therefore

$$\alpha - \beta + 3 = 0, 2\alpha - \beta + 1 = 0.$$

Solving these equations, we get $\alpha = 2, \beta = 5$. These are the required values of α and β respectively.

Example 1.5.5

To what point must be origin be removed so that first degree terms vanish of the equation $2x^2 - 3y^2 - 4x - 12y = 0$.

Solution: Lt us consider the origin is shifted to the point (α, β), then

$$x = x' + \alpha, \quad y = y' + \beta.$$

The given equation is $2x^2 - 3y^2 - 4x - 12y = 0$.Then

$$\begin{aligned}
&2\,(x' + \alpha)^2 - 3\,(y' + \beta)^2 - 4\,(x' + \alpha) - 12\,(y' + \beta) = 0\\
\Rightarrow &2x'^2 - 3y'^2 + (4\alpha - 4)x' - (6\beta + 12)y' + \left(2\alpha^2 - 4\alpha\right) - \left(3\beta^2 + 12\beta\right) = 0
\end{aligned}$$

Since 1^{st} degree terms vanish, therefore $4\alpha - 4 = 0$ and $6\beta + 12 = 0$.
Solving these equations, we get $\alpha = 1, \beta = -2$. Hence the required point is $(1, -2)$.

Example 1.5.6

Find the transformed the equation of the line $y = \sqrt{3}x$. when the axes are rotated through an angle $\frac{\pi}{3}$ in the positive sense.

Solution: Here $\frac{\pi}{3}$ is the angle of rotation. Then

$$x = X\cos\frac{\pi}{3} - Y\sin\frac{\pi}{3} = \frac{X - \sqrt{3}Y}{2};$$

$$y = X\sin\frac{\pi}{3} + Y\cos\frac{\pi}{3} = \frac{\sqrt{3}X + Y}{2}.$$

The given line is

$$y = \sqrt{3}x \Rightarrow \frac{\sqrt{3}X + Y}{2} = \sqrt{3}\left(\frac{X - \sqrt{3}Y}{2}\right)$$

$$\Rightarrow Y = -3Y \Rightarrow 4Y = 0 \Rightarrow Y = 0$$

This is the transformed form of the given equation.

Example 1.5.7

What will be the form of equation $x^2 - y^2 = 4$ if the coordinate axes are roteved through an angle $-\frac{\pi}{2}$?

Solution: Here $-\frac{\pi}{2}$ is the angle of rotation. Then

$$x = X\cos\left(-\frac{\pi}{2}\right) - Y\sin\left(-\frac{\pi}{2}\right) = Y;$$
$$y = X\sin\left(-\frac{\pi}{2}\right) + Y\cos\left(-\frac{\pi}{2}\right) = -X$$

The given line is $x^2 - y^2 = 4 \Rightarrow Y^2 - (-X)^2 = 4 \Rightarrow Y^2 - X^2 = 4$ This is the transformed form of the given equation.

Example 1.5.8

If under rotation an expression of the form $ax + by$ changes $0AX + BY$, show that $a^2 + b^2$ remains invariant.

Solution: Let θ be the angle of rotation. Then

$$\begin{aligned}
&x = X\cos\theta - Y\sin\theta; \quad y = X\sin\theta + Y\cos\theta\\
&ax + by = a(X\cos\theta - Y\sin\theta) + b(X\sin\theta + Y\cos\theta)\\
&\qquad\qquad = (a\cos\theta + b\sin\theta)X + (-a\sin\theta + b\cos\theta)Y\\
&\qquad\qquad = AX + BY(\text{ given })\\
&\therefore A = a\cos\theta + b\sin\theta\\
&B = -a\sin\theta + b\cos\theta
\end{aligned}$$

and Squaring and adding (1) and (2), we get

$$\begin{aligned}
A^2 + B^2 &= (a\cos\theta + b\sin\theta)^2 + (-a\sin\theta + b\cos\theta)^2\\
&= a^2\left(\cos^2\theta + \sin^2\theta\right) + b^2\left(\cos^2\theta + \sin^2\theta\right) = a^2 + b^2
\end{aligned}$$

Thus $a^2 + b^2$ remains invariant.

Example 1.5.9

Show that the equation $x^2 + y^2 = a^2$ is an invariant under rotation of axes.

Solution: Let θ be the angle of rotation. Then

$$x = x'\cos\theta - y'\sin\theta; \; y = x'\sin\theta + y'\cos\theta$$

We have

$$\begin{aligned}
&x^2 + y^2 = a^2\\
&\Rightarrow (x'\cos\theta - y'\sin\theta)^2 + (x'\sin\theta + y'\cos\theta)^2 = a^2\\
&\Rightarrow \left(\cos^2\theta + \sin^2\theta\right)x'^2 + \left(\cos^2\theta + \sin^2\theta\right)y'^2 = a^2\\
&\Rightarrow x'^2 + y'^2 = a^2
\end{aligned}$$

Thus the equation $x^2 + y^2 = a^2$ is an invariant under rotation of axes.

Example 1.5.10

Find the coordinates of the point (-2,4) referred to new axes obtained by rotating the old axes through an angle of 45° in the positive sense.

Solution: Here 45° is the angle of rotation. Then

$$x' = x\cos 45^\circ + y\sin 45^\circ = -\frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \sqrt{2}, (\because x = -2 \& y = 4);$$
$$y' = -x\sin 45^\circ + y\cos 45^\circ = \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}} = 3\sqrt{2}$$

Thus $(\sqrt{2}, 3\sqrt{2})$ is the transformed form of the given point $(-2, 4)$.

Example 1.5.11

Find the transformed form of the equation $x^2 - y^2 = a^2$ when the axes are turn through an angle of 45° keeping the origin fixed.

Solution: Here 45° is the angle of rotation. Then

$$x = X\cos 45^\circ - Y\sin 45^\circ = \frac{X-Y}{\sqrt{2}};$$
$$y = X\sin 45^\circ + Y\cos 45^\circ = \frac{X+Y}{\sqrt{2}}$$

The given equation is $x^2 - y^2 = a^2$

$$\Rightarrow \left(\frac{X-Y}{\sqrt{2}}\right)^2 - \left(\frac{X+Y}{\sqrt{2}}\right)^2 = a^2$$
$$\Rightarrow -\frac{4XY}{2} - a^2 = 0 \Rightarrow 2XY + a^2 = 0$$

This is the transformed form of the given equation.

Example 1.5.12

What does the equation $11x^2 + 16xy - y^2 = 0$ take the form on turning the axes through an angle $\tan^{-1}\frac{1}{2}$.

Solution: Here $\tan^{-1}\frac{1}{2} = \theta$ (say) is the angle of rotation. Then

$$\sin\theta = \frac{2}{\sqrt{5}}, \cos = \frac{1}{\sqrt{5}}$$
$$\therefore \quad x = X\cos\theta - Y\sin\theta = \frac{2X-Y}{\sqrt{5}}$$

and

$$y = X\sin\theta + Y\cos\theta = \frac{X+2Y}{\sqrt{5}}.$$

The given equation is $11x^2 + 16xy - y^2 = 0$

$$\Rightarrow 11\left(\frac{2X-Y}{\sqrt{5}}\right)^2 + 16\left(\frac{2X-Y}{\sqrt{5}}\right)\left(\frac{X+2Y}{\sqrt{5}}\right) - \left(\frac{X+2Y}{\sqrt{5}}\right)^2 = 0$$
$$\Rightarrow 11\left(4X^2 - 4XY + Y^2\right) + 16\left(2X^2 + 3XY - 2Y^2\right) - \left(X^2 + 4XY + 4Y^2\right) = 0$$
$$\Rightarrow 11X^2 - 25Y^2 = 0$$

This is the transformed form of the given equation.

Example 1.5.13

Find the angle through which the axes are to be rotated so that the equation $x\sqrt{3} + y + 6 = 0$ may be reduced to $x = c$. Also, determine the value of c.

Solution: Let the axes be rotated through an angle θ. Then

$$x = x'\cos\theta - y'\sin\theta \& y = x'\sin\theta + y'\cos\theta$$

The given equation is $x\sqrt{3} + y + 6 = 0$. Then the reduced equation becomes

$$\sqrt{3}\,(x'\cos\theta - y'\sin\theta) + (x'\sin\theta + y'\cos\theta) + 6 = 0$$
$$\Rightarrow x'(\sqrt{3}\cos\theta + \sin\theta) + y'(\cos\theta - \sqrt{3}\sin\theta) + 6 = 0$$

Since this is to be of the form $x = c$, then

$$\cos\theta - \sqrt{3}\sin\theta = 0 \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$$

Thus the transformed equation is, after removing the primes

$$x\left(\sqrt{3}\times\frac{\sqrt{3}}{2} + \frac{1}{2}\right) + y(0) + 6 = 0 \Rightarrow 2x + 6 = 0 \Rightarrow x = -3$$

Hence the required value of c is -3.

Example 1.5.14

Find the transformed equation of the curve

$$(3x + 4y + 7)(4x - 3y + 5) = 50$$

when the axis are $3x + 4y + 7 = 0$ and $4x - 3y + 5 = 0$.

Solution: Let (x', y') be the coordinates of a point (x, y) referred to the new set of axes. Then

$$y' = \frac{3x + 4y + 7}{\sqrt{3^2 + 4^2}} = \frac{3x + 4y + 7}{5} \text{ and } x' = \frac{4x - 3y + 5}{\sqrt{4^2 + 3^2}} = \frac{4x - 3y + 5}{5}.$$

The given equation can be written as

$$\frac{3x + 4y + 7}{5}\cdot\frac{4x - 3y + 5}{5} = 2 \Rightarrow x'y' = 2.$$

Hence the required transformed equation is $xy = 2$.

Example 1.5.15

Transform the equation of the curve

$$3(12x - 5y + 39)^2 + 2(5x - 12y - 26)^2 = 169$$

taking the lines $12x - 5y + 39 = 0$ and $5x - 12y - 26 = 0$ are the new axes of x and y respectively.

Solution: Let (x', y') be the coordinates of a point (x, y) referred to the new set of axes. Then

$$y' = \frac{12x - 5y + 39}{\sqrt{12^2 + 5^2}} = \frac{12x - 5y + 39}{13} \text{ and } x' = \frac{5x - 12y - 26}{\sqrt{5^2 + 12^2}} = \frac{5x - 12y - 26}{13}.$$

The given equation can be written as

$$3\,(13y')^2 + 2\,(13x')^2 = 169 \text{ or } 2x'^2 + 3y'^2 = 1.$$

Hence the required transformed equation is $2x^2 + 3y^2 = 1$.

Example 1.5.16

Show that the equation $4xy - 3x^2 = 0$ is transformed to $x'^2 - 4y'^2 = 1$ by rotating the axes through an angle $\tan^{-1} 2$.

Solution: Here $\tan^{-1} 2$ is the rotation angle. Then

$$x = x'\cos\theta - y'\sin\theta = \frac{x' - 2y'}{\sqrt{5}};$$
$$y = x'\sin\theta + y'\cos\theta = \frac{2x' + y'}{\sqrt{5}}, \text{ where } \theta = \tan^{-1} 2$$

The given equation is $4xy - 3x^2 = 0$

$$\Rightarrow 4\left(\frac{x' - 2y'}{\sqrt{5}}\right)\left(\frac{2x' + y'}{\sqrt{5}}\right) - 3\left(\frac{x' - 2y'}{\sqrt{5}}\right)^2 = 0$$
$$\Rightarrow 4\left(2x'^2 - 3x'y' - 2y'\right) - 3\left(x'^2 - 4x'y' + 4y'^2\right) = 0$$
$$\Rightarrow x'^2 - 4y'' = 0$$

This is the transformed form of the given equation.

Example 1.5.17

To what point the origin is to be moved so that one can get rid of the first-degree term from the equation

$$x^2 + xy + 2y^2 - 7x - 5y + 12 = 0 \text{ ?}$$

Solution: Let the point to which the origin will be shifted be (α, β).
Substituting $x = x' + \alpha, y - y' + \beta$, the equation becomes

$$(x' + \alpha)^2 + (x' + \alpha)(y' + \beta) + 2(y' + \beta)^2 - 7(x' + \alpha) - 5(y' + \beta) + 12 = 0.$$

The coefficients of x' and y' in the transformed equation are

$$(2\alpha + \beta - 7) \text{ and } (\alpha + 4\beta - 5),$$

which will be separately zero, if the first degree terms are to be removed.
Thus $2\alpha + \beta - 7 = 0$ and $\alpha + 4\beta - 5 = 0$.
Solving, we get $\alpha = 3\frac{2}{7}, \beta = \frac{3}{7}$.
Hence the origin must be shifted to the point $\left(3\frac{2}{7}, \frac{3}{7}\right)$.

Example 1.5.18

Find the angle through which the axes are to be rotated so that the equation $x\sqrt{3} + y + 6 = 0$ may be reduced to $x - c$. Also, determine the value of c.

Solution: Let the axes be rotated through an angle θ, so that

$$x = x'\cos\theta - y'\sin\theta \text{ and } y = x'\sin\theta + y'\cos\theta.$$

The reduced equation becomes, after removing the primes,

$$\sqrt{3}(x\cos\theta - y\sin\theta) + (x\sin\theta + y\cos\theta) + 6 = 0$$

or, $x(\sqrt{3}\cos\theta + \sin\theta) + y(\cos\theta - \sqrt{3}\sin\theta) + 6 = 0$. Since this is to be of the form $x = c$, therefore

$$\cos\theta - \sqrt{3}\sin\theta - 0, \text{ whence } \theta - \frac{1}{6}\pi.$$

Thus the transformed equation is

$$x\left(\frac{3}{2} + \frac{1}{2}\right) = -6$$
$$\text{or,} \quad x = -3$$
$$\text{Hence} \quad c = -3.$$

Example 1.5.19

Find the angle by which the axes should be rotated so that the equation $ax^2 + 2hxy + by^2 = 0$ becomes another equation in which the term xy is absent.
In particular, find the angle through which the axes are to be rotated so that the equation

$$17x^2 + 18xy - 7y^2 = 1$$

may be reduced to the form $Ax^2 + By^2 = 1, A > 0$; find also A and B.

Solution: Let the axes be turned through an angle θ. Then, after removing the primes from

(x', y'), the general form of the equation becomes

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + b(x\sin\theta + y\cos\theta)^2 = 0.$$

The coefficient of xy will be zero, if

$$(b-a)\sin\theta\cos\theta + h\left(\cos^2\theta - \sin^2\theta\right) = 0,$$

that is,

$$-\frac{1}{2}(a-b)\sin 2\theta + h\cos 2\theta = 0,$$

that is,

$$\tan 2\theta = \frac{2h}{a-b}, \text{ giving } \theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}.$$

In the given case, $a = 17, h = 9, b = -7$. Therefore $\theta = \frac{1}{2}\tan^{-1}\frac{18}{17+7} = \frac{1}{2}\tan^{-1}\frac{3}{4}$. By invariant, we get

$$A + B = 17 - 7 = 10 \text{ and}$$

$$AB = 17(-7) - 9^2 = -200$$

Therefore $(A-B)^2 = (A+B)^2 - 4AB = 900$

$$\text{or,} \quad A - B = \pm 30.$$

Hence $A = 20, B = -10 (\because A > 0)$, and the transformed equation is $20x^2 - 10y^2 = 1$.